

HALF DIMENSIONAL COLLAPSE OF ENDS OF MANIFOLDS OF NONPOSITIVE CURVATURE

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ABSTRACT. We study the topology of ends of noncompact, complete Riemannian n -manifolds M with bounded nonpositive sectional curvature and finite volume. It is known that if such a manifold is negatively curved or does not contain arbitrarily small geodesic loops, then it is tame in the sense that it is homeomorphic to the interior of a compact manifold \overline{M} with boundary $\partial\overline{M}$ since the thin part $M_{<\epsilon}$, i.e. the end of M , is topologically a product of a closed manifold with a ray. Let $\widetilde{M}_{<\epsilon}$ be a lift of $M_{<\epsilon}$ in the universal cover \widetilde{M} . We show that in this case, any finite polyhedron P in $\widetilde{M}_{<\epsilon}$ can be homotoped within $\widetilde{M}_{<\epsilon}$ to factor through a polyhedron Q of dimension less than $\lfloor n/2 \rfloor$. A corollary of this is that the homology of $\widetilde{M}_{<\epsilon}$ vanishes in dimension greater or equal to $\lfloor n/2 \rfloor$. Another corollary is that when M has dimension less than 6 each component of the boundary $\partial\overline{M}$ is aspherical. A third corollary is that any complex homotopy equivalent to M has dimension greater or equal to $\lceil n/2 \rceil$. These bounds are sharp by examples such as products of noncompact hyperbolic surfaces.

1. INTRODUCTION

Let M be a noncompact, complete Riemannian n -manifold with bounded nonpositive sectional curvature $-1 < K \leq 0$ and finite volume¹. We assume that M does not have arbitrarily small geodesic loops. This is true when M is negatively curved by a theorem of Gromov ([8]). It follows in the same way as in the argument in Gromov's paper that because there are no small geodesic loops, the injectivity radius function on M does not have any critical point outside a compact set which can be taken to be the thick part $M_{>\epsilon}$ for some *small* $\epsilon > 0$ (see also Appendix 2 of [4] for a generalization). Therefore, M is *tame* in the sense that the thin part $M_{<\epsilon}$ has finitely many components and each component is topologically a product of a closed $(n-1)$ -manifold with a ray. Let $\widetilde{M}_{<\epsilon}$ be a lift of the thin part $M_{<\epsilon}$ in the universal cover \widetilde{M} . We will call $\widetilde{M}_{<\epsilon}$ the thin part of \widetilde{M} . It is the topology of $\widetilde{M}_{<\epsilon}$ that we would like to describe.

Good examples to think about are locally symmetric spaces of noncompact type, such as hyperbolic manifolds, products of surfaces, and the usual beloved $K \backslash G / \Gamma$. Sometimes taking $G = \mathrm{SL}_m \mathbb{R}$ and $K = \mathrm{SO}_m$ can be as

¹In fact, all our results hold with “finite volume” replaced by “injectivity radius $\rightarrow 0$ ”.

satisfactory as any other semisimple Lie groups. This sentiment holds true in terms of examples to keep in mind as one reads this paper since our approach to understanding the topology of the end of M is purely geometric/topological but can be demonstrated by thinking about these concrete examples the right way.

When M is locally symmetric, $\widetilde{M}_{<\epsilon}$ is homotopy equivalent to the spherical Tits building at infinity, which is a $(k-1)$ -dimensional complex, where k is the (rational) rank of M . The Tits building is homotopy equivalent to a wedge of $(k-1)$ -dimensional spheres ([6]). The rank k is at most $n/2$ with $k = n/2$ when M is a product of non-compact surfaces. The main purpose of this paper is to show that this is no arithmetic coincidence but in a slightly weaker sense. We prove that in the general nonpositively curved setting, for an n -dimensional manifold M satisfying the conditions described above, any polyhedron in $\widetilde{M}_{<\epsilon}$ can be homotoped within $\widetilde{M}_{<\epsilon}$ to one with dimension at most $\lfloor n/2 \rfloor - 1$, where $\lfloor n/2 \rfloor$ is the largest integer less than or equal to $n/2$.

Theorem 1. *Let M be a noncompact, complete, Riemannian manifold with bounded nonpositive sectional curvature $-1 < K \leq 0$ and finite volume. Assume that M has no arbitrarily small geodesic loops. Let $\epsilon > 0$ be small enough so that $M_{<\epsilon}$ is a topologically a product with a ray.*

Let P be a finite polyhedron and let $\varphi: P \rightarrow \widetilde{M}_{<\epsilon}$ be a continuous map. Then φ can be homotoped within $\widetilde{M}_{<\epsilon}$ to a map $\widehat{\varphi}: P \rightarrow \widetilde{M}_{<\epsilon}$ such that $\widehat{\varphi}$ factors through a polyhedron Q of dimension less than $\lfloor n/2 \rfloor$. That is, $\widehat{\varphi} = \tau \circ \pi$, for

$$P \xrightarrow{\pi} Q \xrightarrow{\tau} \widetilde{M}_{<\epsilon}.$$

The following two corollaries are immediate consequences.

Corollary 2. *When M has dimension ≤ 5 , the thin part $M_{<\epsilon}$ is aspherical.*

Corollary 3. *The homology of $\widetilde{M}_{<\epsilon}$ vanishes in dimension $\geq \lfloor n/2 \rfloor$, i.e.*

$$(1) \quad H_{\geq \lfloor n/2 \rfloor}(\widetilde{M}_{<\epsilon}) = 0.$$

Remark. The upper bound on the dimension of Q is sharp by the following example. If M is the product of k hyperbolic punctured tori, then M has dimension $n = 2k$, so $\lfloor n/2 \rfloor - 1 = k - 1$. We also know that in this case $\widetilde{M}_{<\epsilon}$ is homotopy equivalent to a wedge of $(k-1)$ -spheres. In a forthcoming paper [3] we build, for each n , an n -manifold M satisfying the hypotheses of Theorem 1 which satisfies $\overline{H}_k(\widetilde{M}_{<\epsilon}) \neq 0$ for all $k < \lfloor n/2 \rfloor$. So, unlike in the case of locally symmetric spaces, in general there are no low dimensional homology vanishing results complementing the high dimensional homology vanishing (1) and in general $\pi_1 M$ is not a duality group.

The vanishing of high dimensional homology (1) is equivalent to the vanishing of low dimensional group cohomology with group ring coefficients

$$(2) \quad H^{<[n/2]}(B\Gamma; \mathbb{Z}\Gamma) = 0.$$

Another consequence of (1) is that the geometric dimension² of Γ , denoted $\text{gdim}(\Gamma)$, is at least $[n/2]$, where $[n/2]$ denotes the smallest integer $\geq n/2$.

Corollary 4. *The geometric dimension of Γ is $\geq [n/2]$,*

In the extreme case when $\text{gdim}(\Gamma)$ is precisely equal to $[n/2]$ we get more.

Corollary 5. *If $\text{gdim}(\Gamma) = [n/2]$ then*

- Γ is a duality group³, and
- Γ contains a rank $[n/2]$ free abelian group of isometries.

Remark. We obtain stronger results in the special case when the largest rank r of abelian subgroups of Γ is strictly smaller than $[n/2]$. In this case, Theorem 1 and its corollaries hold true with $[n/2]$ replaced by r and $[n/2]$ replaced by $n - r$. On the other hand, it is possible for Γ to have abelian subgroups of high rank. When M is a noncompact, finite volume hyperbolic n -manifold, then the fundamental group Γ contains a parabolic abelian subgroup of rank $n - 1$, which is much bigger than $[n/2]$.

The raw statement that we actually prove is Theorem 6 below, which is a little harder to motivate than Theorem 1. However, Theorem 1 follows with not much work from Theorem 6.

Theorem 6. *Assume the hypotheses of Theorem 1. Let $P^{([n/2]-1)}$ be the $([n/2] - 1)$ -skeleton of P . Then φ can be homotoped within $\widetilde{M}_{<\epsilon}$ to a Lipschitz map $\overline{\varphi}: P \rightarrow \widetilde{M}_{<\epsilon}$ such that the image $\overline{\varphi}(P)$ is the same as $\overline{\varphi}(P^{([n/2]-1)})$.*

The *intuition* (which is not to be confused with *motivation*) behind the proof of the above theorem is that we push the polyhedron P to infinity within $\widetilde{M}_{<\epsilon}$ without being too stupid in the way we push it. Note that we can always push anything in $\widetilde{M}_{<\epsilon}$ to infinity since M has tame ends, but we want to push P in such a way not to stretch it more than we absolutely have to. The number of degrees of freedom in stretching P is the dimension of the complex that we can collapse P to, which we will show to have dimension less than $[n/2]$.

We motivate the proof of Theorem 6 in Section 2, in which we try to explain why things are done the way they are through an iteration of “what is

²The *geometric dimension* of Γ is the minimum dimension of an aspherical complex $B\Gamma$ with fundamental group Γ .

³A group Γ is a *duality group* if $H^*(B\Gamma; \mathbb{Z}\Gamma)$ is concentrated in a single dimension.

the simplest thing to do?” and “what are the problems to overcome?” until there are no more problems. The rest of the paper builds up the machinery for the proofs of the main theorems, which emerge in Section 10 of the paper. In time of troubles, good things to think about are locally symmetric spaces, in particular the examples of $\mathrm{SO}_3 \setminus \mathrm{SL}_3 \mathbb{R} / \mathrm{SL}_3 \mathbb{Z}$ and products of surfaces.

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2. PROBLEMS AND SOLUTIONS

Let $\epsilon > 0$ be smaller than both the Margulis constant and the smallest geodesic loop in M so that at each point $x \in M_{<\epsilon}$ the injectivity radius at x is realized by a parabolic isometry of \widetilde{M} . It follows that the injectivity radius function has no critical points on $M_{<\epsilon}$ so that M is tame.

Let P be as in Theorem 1. One can naively take a length minimizing geodesic ray γ in M starting at a point in $\varphi(P)$, and take a lift $\tilde{\gamma}$ of γ , and then push P toward $\tilde{\gamma}(\infty) \in \partial_\infty \widetilde{M}$ with unit speed. Then the diameter of P will stay bounded, so once it is far enough to infinity it will be contained in a ball that is contained in $\widetilde{M}_{<\epsilon}$. We then can contract P to a point within this ball. However, there is a problem with this approach, which is that as we push P toward $\tilde{\gamma}(\infty)$ it might slide off $\widetilde{M}_{<\epsilon}$ for some time during this process. This problem does not have a solution for otherwise one can contract any such P to a point within $\widetilde{M}_{<\epsilon}$, which is not true for products of noncompact surfaces.

So we need to find a way to push P to infinity without it leaving $\widetilde{M}_{<\epsilon}$. The strategy is that we push different points of P to different points in the visual boundary $\partial_\infty \widetilde{M}$ along geodesics and we keep track of the amount of directions to infinity we need. The set of points in $\partial_\infty \widetilde{M}$ to which we push P tells us how much P “expands” as we push it to infinity. It also gives a complex in $\widetilde{M}_{<\epsilon}$ to which we can “collapse” P onto. We then bound the dimension of this complex to be less than $\lfloor n/2 \rfloor$ by trying to make this process as efficient (in terms of how many degrees of freedom are needed as P expands) as possible.

Keeping the homotopy within the thin part $\widetilde{M}_{<\epsilon}$. For each point x of the polyhedron P , we find in $\partial_\infty \widetilde{M}$ a point $\rho(x)$ to which we push x with unit speed along the geodesic connecting x with $\rho(x)$ as illustrated in Figure 1. We will define $\rho: P \rightarrow \partial_\infty \widetilde{M}$ systematically, skeleton by skeleton. We

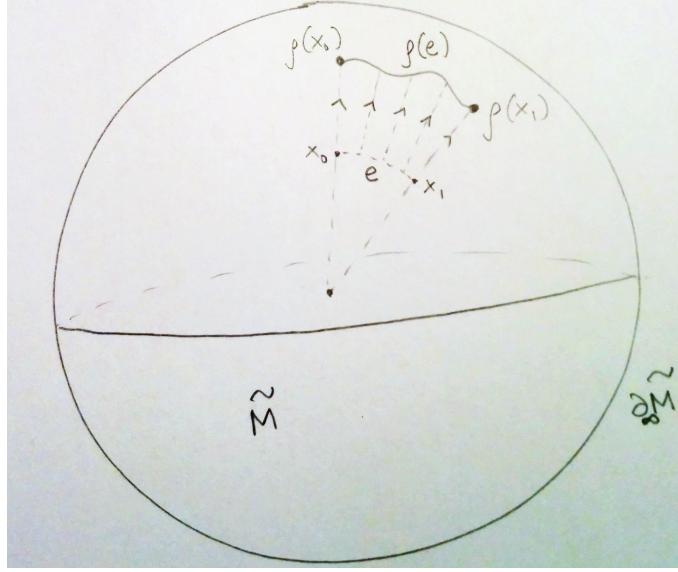


FIGURE 1.

call this homotopy

$$\varphi_t: P \rightarrow \widetilde{M}_{<\epsilon}, \quad \text{for } t \in [0, \infty).$$

To be safe, we will assume that $\varphi: P \rightarrow \widetilde{M}_{<\epsilon}$ for some positive $\epsilon < \epsilon$ to be determined later.

Start with the vertices of P and suppose that x is a vertex of P . Since $\varphi(x)$ is in $\widetilde{M}_{<\epsilon}$, there is a parabolic isometry γ_x that moves $\varphi(x)$ by a *small*⁴ amount, where *small* means less than ϵ , so that the group generated by such γ_x is virtually nilpotent by the Margulis lemma. So a natural choice⁵ for $\rho(x)$ is the center of a horosphere preserved by a nilpotent⁶ group Γ_x generated by parabolic isometries that are *small* at $\varphi(x)$ because if we push x to $\rho(x)$, the *small* elements in Γ_x will remain *small* along the trajectory $\varphi_t(x)$ and therefore $\varphi_t(x)$ will stay in $\widetilde{M}_{<\epsilon}$.

Next, we extend ρ to the edges and higher dimensional simplices of P . Just like for the vertices, the way to go to infinity is to “follow the shrinking small loops”. Let e be the edge connecting vertices x_0 and x_1 of P . Clearly a problem is that there is no clear “transition” in terms of *small* loops at

⁴*Small* can be taken to be less than ϵ here. However, for things to work later we will need to make *small* smaller. So let *small* mean less than $\epsilon/(3^{n+1}I_n!)$, where I_n is the upper bound on the index of the nilpotent subgroup in the Margulis lemma.

⁵Note that the choice of $\rho(x)$ may not be unique. Think about products of surfaces.

⁶The group Γ_x is *virtually* nilpotent by the Margulis lemma but we can drop the virtuality of nilpotency of Γ_x by taking a finite index subgroup of Γ_x as in Section 3.

$\varphi(x_0)$ to *small* loops at $\varphi(x_1)$. But there is a solution, which is to take a fine enough subdivision of P at the beginning. We take a subdivision of P in which the diameter of each simplex is *small*, so that if σ is a k -simplex of P with vertices x_0, x_1, \dots, x_k , then for each point $y \in \sigma$, the *small* parabolic isometries at each $\varphi(x_i)$ are *smallish*⁷ at $\varphi(y)$, where *smallish* means less than ϵ . We can always take such a subdivision of P , so we can harmlessly assume that P is triangulated in such a way at the beginning.

This gives us a way to assign to each simplex a nilpotent group generated by *smallish* loops as follows. Let $\Gamma = \pi_1(M)$. For a vertex x of P , let $S_x = \{\gamma \in \Gamma \mid d(x, \gamma(x)) \text{ is small}\}$ be the set of *small* parabolic isometries at $\varphi(x)$. We assign to x the nilpotent group

$$\Gamma_x = \langle \gamma^{I_n!} \mid \gamma \in S_x \rangle.$$

If $\sigma = x_0 * x_1 * \dots * x_k$ is a k -simplex, then we assign to σ the nilpotent group generated by Γ_{x_i} , for $i = 0, 1, \dots, k$,

$$\Gamma_\sigma = \langle \Gamma_{x_0}, \dots, \Gamma_{x_k} \rangle.$$

Note that the group Γ_σ is generated by *smallish* parabolic isometries due to the fine triangulation on P that we picked at the beginning. Let $Z_\sigma = Z_{\Gamma_\sigma}$ be the center of Γ_σ for later use.

Now the problem with this is that this does not help with defining ρ on the edge e because what we get from Γ_e is the center of a horosphere preserved by Γ_e to which we can push e without it leaving $\widetilde{M}_{<\epsilon}$. However, this center is only one point while what we are looking for is a path connecting $\rho(x_0)$ and $\rho(x_1)$. Nevertheless, what we have obtained is a way to define ρ on the vertices of the first barycentric subdivision P_1 of P .

There will be a solution to the above problem if the distance between adjacent vertices of P_1 in the Tits metric Td on $\partial_\infty \widetilde{M}$ is less than π , in which case life would be easy since there will be a unique geodesic in $(\partial_\infty \widetilde{M}, \text{Td})$ connecting them, so we can use this to define ρ on the edges of P_1 .

But life is hard as we know it⁸. There is a problem, which is that this may not be true. However, there is a solution, which is to use the second barycentric subdivision P_2 of P and to replace the above nilpotent groups by certain abelian groups. Since nilpotent groups are at times harder to deal with than abelian groups, we try to make life as easy as we can by assigning to the simplex σ the abelian group Z_σ instead of Γ_σ . Note that Z_σ has *smallish* elements. We assign abelian groups to vertices of P_2 as follows.

⁷*Smallish* means what *small* used to mean before we made *small* smaller.

⁸As Ilya Gekhtman once said, “Life is not unfair. It’s unpleasant.”

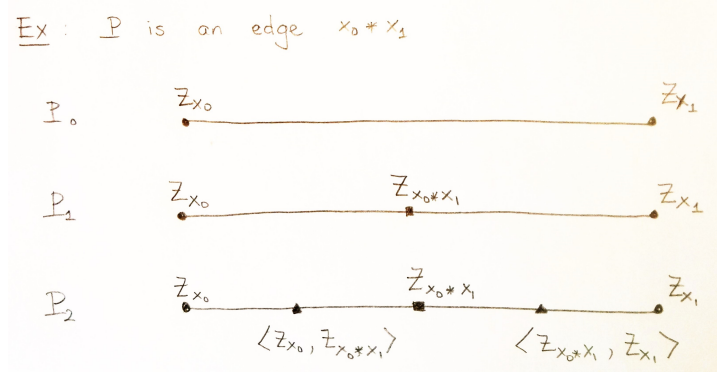


FIGURE 2.

- P_0 -stage: for each vertex x of P , we assign the abelian group Z_x to x . Once this is done, we assign abelian groups to the vertices created in the first barycentric subdivision P_1 of P as follows.
- P_1 -stage: At the vertex corresponding to the interior point of a simplex $\sigma = x_0 * x_1 * \dots * x_k$, we assign the group

$$Z_\sigma = Z_{x_0 * x_1 * \dots * x_k} = Z_{\langle \Gamma_{x_0}, \dots, \Gamma_{x_k} \rangle}.$$

We will call this vertex σ . It is not hard to see that Z_σ contains parabolic isometries that are *smallish* at all points of σ . Once this is done, we move on to the vertices created in P_2 .

- P_2 -stage: At the vertex corresponding to the interior point of a simplex $\Delta_{x_0 x_1 \dots x_k} := (x_0 \subset x_0 * x_1 \subset \dots \subset x_0 * x_1 * \dots * x_k)$ in P_2 , we assign the abelian group generated by all the abelian groups assigned to the vertices of the simplex, that is,

$$A_{\Delta_{x_0 x_1 \dots x_k}} := \langle Z_{x_0}, Z_{x_0 * x_1}, \dots, Z_{x_0 * x_1 * \dots * x_k} \rangle.$$

See Figure 2 and Figure 12.

Warning (switch of notation): We use “A” to denote this abelian group because it is not a center. For simplicity/uniformity of notation, we change all the centers Z_x to A_x since what we will use is the fact that these groups are abelian.

We have assigned to each vertex x in P_2 an abelian group A_x containing parabolic isometries, some of which are *smallish*. With a little bit of thought, we see that for any k -simplex Σ in P_2 there is a vertex with the biggest abelian group, which we will call A_Σ , which is generated by the abelian groups at all the other vertices of the simplex. This turns out to be crucial at more than one place later.

Next, we describe how to show that adjacent vertices in P_2 are within a distance $\pi/2$ in the Tits metric. For an abelian group A_x , where x is a vertex of P_2 , there is a way to find a non-empty set of points in $\partial_\infty \widetilde{M}$, which we will call ∂A_x ⁹, that serves as a “center at infinity” of A_x in the sense that horospheres at each point of ∂A_x are preserved by the centralizer C_{A_x} of A_x . The construction of this “center of infinity” is discussed in Section 4. We also obtain from this construction that for a chain of abelian groups $A_{x_0} \leq A_{x_1} \leq \dots \leq A_{x_k}$, the diameter of the union of $\partial A_{x_0} \cup \partial A_{x_1} \cup \dots \cup \partial A_{x_k}$, with respect to the Tits metric, is less than or equal to $\pi/2$.

With the way abelian groups are assigned to the vertices of P_2 , it turns out that for each k -simplex σ of P_2 , one can write $\sigma = x_0 * x_1 * \dots * x_k$ such that $A_{x_0} \leq A_{x_1} \leq \dots \leq A_{x_k}$. Therefore, for each vertex x of P_2 we can pick a point in ∂A_x and define $\rho(x)$ to be that point, and it will follow that for any two adjacent vertices x_0 and x_1 in P_2 the distance $\text{Td}(\rho(x_0), \rho(x_1)) \leq \pi/2$, so there is a unique geodesic $\rho(e)$ in $\partial_\infty \widetilde{M}$ connecting $\rho(x_0)$ and $\rho(x_1)$. Parametrize both e and $\rho(e)$ by constant speed and use this to define ρ on e the obvious way. We can extend ρ to higher dimensional skeleta via geodesic triangles in the obvious way.

Now that we have found a way to define ρ on P that captures the “transition” of *smallish* loops at adjacent vertices, we need to make sure that the homotopy φ_t does not push P off the thin part $\widetilde{M}_{<\epsilon}$. This is not obvious but not a problem. We can show that the *smallish* parabolic isometries at $\varphi(\sigma)$ remain *smallish* at $\varphi_t(\sigma)$ for all $t > 0$. We explain this for the case of an edge e in P_2 with end points x_0 and x_1 . The general case is very similar.

Recall for $i = 0, 1$, the horospheres centered at $\rho(x_i)$ are preserved by the centralizer $C_{A_{x_i}}$. We can also assume without loss of generality that $A_{x_0} \leq A_{x_1}$ since one of them must be the biggest abelian group. Then A_{x_1} preserves horospheres centered at $\rho(x_0)$ since it centralizes A_{x_0} . Therefore, both $\rho(x_0)$ and $\rho(x_1)$ are fixed by A_{x_1} . It follows that the group A_{x_1} fixes pointwise the unique geodesic τ in $\partial_\infty \widetilde{M}$ connecting $\rho(x_0)$ and $\rho(x_1)$. Therefore, if z is a point in τ , and y is a point in e , then the geodesic ray λ connecting $\varphi(y)$ to z stays in $\widetilde{M}_{<\epsilon}$. This is because there are elements in A_{x_1} that are *smallish* at $\varphi(y)$ that remain *smallish* along λ because A_{x_1} fixes z . Hence, the homotopy φ_t does not move e off $\widetilde{M}_{<\epsilon}$, and we can move on to the next task.

“Collapsing” P within $\widetilde{M}_{<\epsilon}$. Now that we have defined ρ of P and made sure that pushing P to $\rho(P)$ does not leave $\widetilde{M}_{<\epsilon}$, we want to find a copy of

⁹Later we will write $\partial(d_{A^0} + d_{A^+})$ instead of ∂A for reasons that hopefully will become evident.

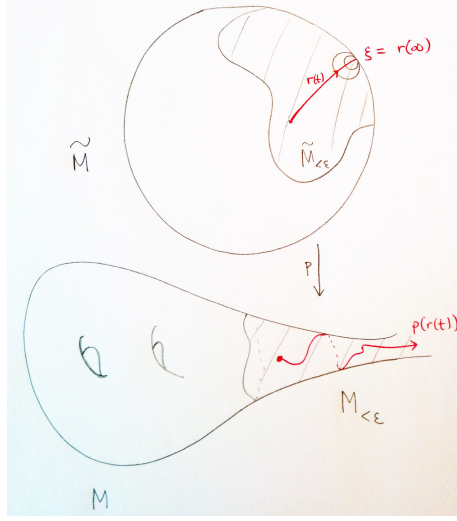


FIGURE 3.

$\rho(P)$ in \tilde{M} to which we can “collapse” P within $\tilde{M}_{<\epsilon}$. Take a point $c_0 \in \tilde{M}$ and take the geodesic cone on $\rho(P)$ with cone point c_0 . For $t \geq 0$ and $x \in P$, let $c_t(x)$ be the point obtained by flowing for time t along the geodesic ray from c_0 to $\rho(x)$. Then $c_t(P)$ homeomorphic to $\rho(P)$ because geodesic retractions are homeomorphisms. Also, it is not hard to see that the distance between $\varphi_t(P)$ and $c_t(P)$ is bounded by some number R that does not depend on $t \geq 0$. We can “collapse” P onto $c_t(P)$ in an R -neighborhood of $\varphi_t(P)$.

However, there is a problem, which is that the “collapse” might leave $\tilde{M}_{<\epsilon}$. In addition to making sure that $\varphi_t(\sigma)$ stays in the thin part $\tilde{M}_{<\epsilon}$ for all $t > 0$, we also need its projection under the covering space projection $p: \tilde{M} \rightarrow M$ to be *divergent* in M . That is, $p(\varphi_t(\sigma))$ leaves all compact sets in M as $t \rightarrow \infty$. This is so that for t large enough, we have enough space to collapse $\varphi_t(P)$ to a lower dimensional complex without having it leave $\tilde{M}_{<\epsilon}$ during the collapse.

This is, indeed, true and the solution is the following key lemma.

Lemma 7 (Divergent Geodesic Ray). *Let A be a free abelian group of isometries. Suppose the centraliser C_A preserves each horosphere centered at a point ξ in $\partial_\infty \tilde{M}$. Then for any geodesic ray $r: [0, \infty) \rightarrow \tilde{M}$ with end point $r(\infty) = \xi$ the projection $p(r(t))$ is divergent.*

More generally, if $r(\infty) = \eta$ for some η fixed by C_A and $\text{Td}(\eta, \xi) < \pi/2$ then $p(r(t))$ is also divergent.

It is useful to introduce the concept of the *thin limit set*, which is the set of all end points $r(\infty)$ of a geodesic ray $r: [0, \infty) \rightarrow \tilde{M}$ such that the

projection $p(r(t))$ is divergent.

This is how we use Lemma 7 to solve the above problem. Suppose we are in the illustrative situation of an edge e in P_2 connecting vertices x_0 to x_1 as above. We again assume without loss of generality that $A_{x_0} \leq A_{x_1}$. In terms of centralizers, we have $C_{A_{x_1}} \leq C_{A_{x_0}}$. By the same argument as above, the group $C_{A_{x_1}}$ also fixes the geodesic τ pointwise. Horospheres at the endpoints of τ are preserved by $C_{A_{x_1}}$ and every point on τ is a distance less than $\pi/2$ from one of the endpoints, so we can apply Lemma 7 to see that each point of τ belongs to the thin limit set. Therefore, for each point y in e , the geodesic ray $\varphi_t(y)$ is divergent. This means that the homotopy φ_t moves the edge e as deep into the thin part $\widetilde{M}_{<\epsilon}$ as we want provided that t is large enough. The same argument works for higher dimensional simplices in P_2 . Once we have moved P so deep into $\widetilde{M}_{<\epsilon}$ that an R -neighborhood of $\varphi_{t_{\text{large}}}(P)$ is contained in $\widetilde{M}_{<\epsilon}$, we can collapse it within the R -neighborhood to the complex $c_{t_{\text{large}}}(P)$.

Bounding the dimension of $\rho(P)$. That the dimension¹⁰ of $\rho(P)$ is at most $\lfloor n/2 \rfloor - 1$ is due to two factors.

- First, each abelian group A_x is the biggest abelian group among those at the vertices of some simplex $\sigma = x_0 * x_1 * \dots * x_k$ in P_2 . So A_x preserves horospheres centered at $\rho(x_i)$ for $i = 0, 1, \dots, k$, and therefore preserves their intersection. If $\rho(x_i)$, for $i = 0, 1, \dots, k$, span an l -dimensional simplex at infinity (for $l \leq k$), then the dimension of the intersection of the horospheres *should* be $n - (l + 1)$. This *should* mean that the rank of A_x is less than or equal to $n - (l + 1)$.
- Second, if σ is a simplex in P_2 , we expect the dimension of $\rho(\sigma)$ to be less than the largest of the ranks of the abelian groups assigned to the vertices of σ . This is because we expect the ranks of the abelian groups at different vertices of σ to be different. The reason is that virtually equivalent abelian groups are too similar to demand different treatments, in particular, they should be assigned the same point at infinity. However, there is a problem because virtually equivalent abelian groups A_x and A_y need not have $\rho(x) = \rho(y)$. But there is a solution because we can modify ρ slightly as discussed below to make sure that they correspond to the same point at infinity.

Putting these two factors together we get that if A_x has rank r , then

$$r \leq n - (l + 1) \quad \text{and} \quad r \geq l + 1.$$

¹⁰By *dimension* here we mean the number of vertices minus one of the biggest non-degenerate simplex. A simplex σ is *nondegenerate* if $\rho(\sigma)$ is not contained in the image $\rho(\partial\sigma)$ of the boundary $\partial\sigma$ of σ .

Therefore, $l + 1 \leq \lfloor n/2 \rfloor$. So the dimension of $\rho(P)$ is at most $\lfloor n/2 \rfloor - 1$.

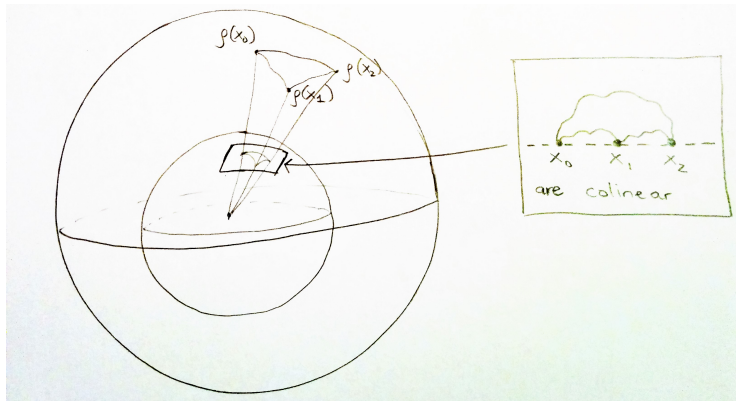
Now we address the problem discussed in the second claim. Recall that for each vertex x of P_2 we pick a point in ∂A_x and define $\rho(x)$ to be that point. This may not be the most efficient way to go to infinity because even with little optimism one expects that if A_x and A_y share a finite index subgroup, there should be a point at infinity whose horospheres are preserved by both A_x and A_y . So if we are stupid enough to pick different points from ∂A_x and ∂A_y for $\rho(x)$ and $\rho(y)$, the complex $\rho(P)$ might be higher dimensional than it needs to be. Another problem is that ∂A_x may not contain *all* points at infinity whose horospheres are preserved by A_x . Fortunately, this problem is not problematic, but it is worth remarking on our human inability to achieve perfection.

The solution to the above problem is to modify ρ slightly as follows. Since P_2 has finitely many vertices, for each vertex x , we take all the groups that virtually contain A_x and take the intersection of all of those and call it $\lfloor A_x \rfloor$. We define ρ exactly the same way as before except that we use $\lfloor A_x \rfloor$ instead of A_x , i.e. we pick a point in $\partial \lfloor A_x \rfloor$ and define $\rho(x)$ to be that point. Since A_x commutes with (or centralizes) $\lfloor A_x \rfloor$, it preserves horospheres centered at each point in $\partial \lfloor A_x \rfloor$. If y is a vertex with A_y virtually equivalent to A_x , then we define $\rho(y) = \rho(x)$. We extend ρ to higher dimensional skeletons of P_2 via geodesics in $\partial_\infty \widetilde{M}$ with the Tits metric as before.

It turns out that there is a problem with the first claim, which is that sometimes things might not be the way they *should*. In this case, it is not clear if the intersection of horospheres described above cannot have dimension larger than $n - (l + 1)$. As unbelievable as it sounds, it is unclear (at least to us) how to rule out the following situation.

Suppose that h_0, h_1 and h_2 are Busemann functions on \widetilde{M} . Let $z_i \in \partial_\infty \widetilde{M}$, for $i = 0, 1, 2$, be the center of the horosphere S_i defined as $h_i = 0$. Now, the intersection $S = S_1 \cap S_2 \cap S_3$ of three horospheres is an $(n - 3)$ -dimensional manifold if S_1, S_2 , and S_3 intersect transversely, i.e. the gradient vectors $\nabla h_0, \nabla h_1$ and ∇h_2 at each point in S are linearly independent. Suppose that z_0, z_1 and z_2 are not co-linear in $\partial_\infty \widetilde{M}$, i.e. none of the three points is on the geodesic connecting the other two, so they span a triangle in $\partial_\infty \widetilde{M}$. Since $\nabla h_0, \nabla h_1$ and ∇h_2 “point toward” z_0, z_1 and z_2 respectively, this strongly suggests that they *should* be linearly independent. However, being linearly independent at a point is too delicate¹¹ a condition and there is no reason to relate the linear structure at a point to what happens at infinity,

¹¹This is not sarcastic.



which is something obtained via a limiting process in terms of the metric. We are not sure if this is a real problem or the problem lies in our inability¹².

But we find a way around this problem and this is a solution, which requires a modification to how ρ is defined. This is the *last* modification we will make to ρ . We define ρ on the vertices of P_2 exactly as we did, but we will not use geodesics to extend ρ to edges and higher dimensional simplices. Instead, we construct what we call *Busemann paths* and *Busemann simplices* and we use them in place of geodesics in the above process of defining ρ . A Busemann simplex with vertices $z_0, z_1, \dots, z_k \in \partial_\infty \widehat{M}$ is a map

$$\sigma: \Delta^k \rightarrow \partial_\infty \widetilde{M}$$

from a k -simplex $x_0 * x_1 * \dots * x_k$ such that $\sigma(x_i) = z_i$ for $i = 0, 1, \dots, k$, and for any point $x \in \Delta^k$, the point $\sigma(x)$ is the “center at infinity” ∂h of a function h that is equal to a convex combination of the Busemann functions h_i centered at z_i , for $i = 0, 1, \dots, k$. As a result of the construction, if a parabolic isometry preserves the horospheres centered at z_i for $i = 0, 1, \dots, k$, then it will preserve horospheres centered at $\sigma(x)$ for each $x \in \Delta^k$.

The incentive for constructing Busemann simplices is to create more points at infinity whose horospheres are preserved so that we can use them in the case when the horospheres centered at the vertices do not intersect transversely. It turns out that even with a whole nondegenerate k -simplex of points at infinity whose horospheres are preserved by an abelian group

¹²It turns out that the case of the intersection of three horospheres is not a problem because linear independence of three vectors is equivalent to a nondegenerate triangle on the unit tangent sphere and being a nondegenerate triangle can be captured by knowing the length of the sides starting at a fixed vertex. A triangle inequality at infinity translates to a triangle inequality in the unit tangent sphere at a point (if we pick the point far enough). However, we cannot resolve the case of the intersection of four or more horospheres because being a nondegenerate tetrahedron cannot be captured by knowing the length of the sides starting at a fixed vertex.

A_x we are unable to even prove existence of $(k+1)$ points whose Busemann functions have linearly independent gradient vectors everywhere. However, Busemann simplices are too good to waste and we use them to show that the first claim is true if geodesic simplices are replaced by Busemann simplices.

Busemann simplices are constructed as limits of simplices $\sigma_{R_i}(\Delta^k)$ on larger and larger spheres centered at some fixed point. We pick such a simplex $\sigma_R(\Delta^k)$ on a large enough sphere to “approximate” the Busemann simplex at infinity well enough so that nondegeneracy at infinity implies nondegeneracy of $\sigma_R(\Delta^k)$. Take the “Busemann cone” C on $\sigma_R(\Delta^k)$ with cone point x . Each point in C belongs to the intersection of some horospheres centered at $z_i = \rho(x_i)$, for $i = 0, 1, \dots, k$. We show that if an abelian group A_x of rank r preserves horospheres centered at z_i , then when we line up these intersections of horospheres over C , the union has dimension at least $r + k + 1$. This gives the first claim.

The fact that Busemann simplices are Lipschitz comes out of the construction, which is discussed in Section 7. The Lipschitz condition is needed in the proof Theorem 1. The reason is that we need σ not to be space filling so that we can use an old trick to deform $\widehat{\varphi}$ so that it factors through a lower dimensional complex.

There are no more problems.

3. THE SETUP AND THE CONSTANTS

Setup. In the rest of the paper, M is a complete, finite volume n -dimensional manifold of bounded non-positive curvature ($-1 \leq K \leq 0$) with fundamental group Γ and universal cover $p : \widetilde{M} \rightarrow M$. Moreover, we assume that there are no arbitrarily small geodesic loops.

Margulis lemma. This lemma (see [4]) says there are constants μ_n and I_n , depending only on the dimension n , so that the group $\langle \gamma \in \Gamma \mid d(x, \gamma x) < \mu_n \rangle$ generated by the elements that move x less than μ_n is virtually nilpotent, and contains a nilpotent subgroup of index $\leq I_n$. The μ_n is called the Margulis constant.

Smallish ϵ . We fix a constant ϵ to be less than the Margulis constant and the length of the smallest geodesic loop in M . Then elements $\gamma \in \Gamma$ which have displacement $< \epsilon$ at some point are parabolic. Also, the ϵ -thin part

$$(3) \quad \widetilde{M}_{<\epsilon} := \{x \in \widetilde{M} \mid d(x, \gamma x) < \epsilon \text{ for some } \gamma \in \Gamma \setminus \{1\}\}$$

is a product $\partial \widetilde{M}_{<\epsilon} \times [0, \infty)$. (See [8, 4]).

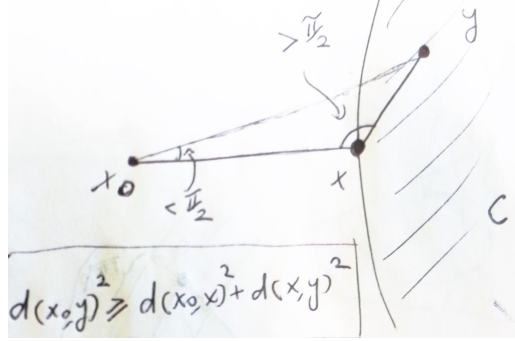


FIGURE 5.

Small ε . Next we fix an even smaller constant

$$(4) \quad \varepsilon := \frac{\epsilon}{3^{n+1}(I_n)!}.$$

Let

$$(5) \quad S_x := \{\gamma \in \Gamma \mid d_\gamma(x) < \varepsilon\},$$

$$(6) \quad \Gamma_x := \langle \gamma^{I_n!} \mid \gamma \in S_x \rangle.$$

Now, if $\sigma = x_0 * \dots * x_k$ is a k -simplex in \widetilde{M} of diameter $< \varepsilon$, then at any point $x \in \sigma$, all the elements $\gamma \in \cup_i S_{x_i}$ have displacement $< 3\varepsilon$. This is less than the Margulis constant, so the group $\langle S_{x_0}, \dots, S_{x_k} \rangle$ is virtually nilpotent, and the group

$$(7) \quad \Gamma_\sigma := \langle \Gamma_{x_0}, \dots, \Gamma_{x_k} \rangle$$

is actually nilpotent. Let Z_σ be its center. It contains an element ζ that can be expressed as a product of at most $3^n(I_n)!$ elements in $\cup_i S_{x_i}$ (see Lemma 8 of [1]). So, this ζ has displacement $\leq 3^n(I_n)!(3\varepsilon) = \epsilon$ everywhere on σ . Note, in particular, that ζ is parabolic because of how we defined ϵ . The choice of these constants will be important in Section 10.

Next, we will see what kind of points the groups Γ_σ, Z_σ etc. fix at infinity.

4. FINDING FIXED POINTS AND INVARIANT HOROSPHERES

4.1. Closest point projection to a convex set. We recall some basic properties of closest point projections in non-positively curved geometry. Let $C \subset \widetilde{M}$ be a convex set and $p_C : \widetilde{M} \rightarrow C$ the closest point projection. Let $x_0 \notin C, y \in C$ and let $x = p_C(x_0)$ be the closest point to x_0 in C . Since C is convex and x is the closest point to x_0 in C , we see (Figure 5) that

$$(8) \quad \angle_x(x_0, y) \geq \pi/2.$$

Because of this, the Euclidean triangle with sides $[x_0, x]$ and $[x, y]$ meeting at an angle $\angle_x(x_0, y)$ has $d(x_0, y)^2 \geq d(x_0, x)^2 + d(x, y)^2$. Triangle comparison

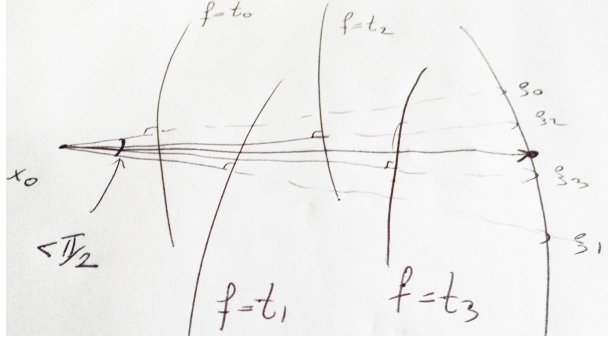


FIGURE 6.

implies that the same is true for our triangle in \widetilde{M} . Consequently,

$$(9) \quad d(x, y) \leq \sqrt{d(x_0, y)^2 - d(x_0, x)^2}.$$

The obtuseness (8) also implies that

$$(10) \quad \angle_{x_0}(x, y) < \pi/2.$$

4.2. Points at infinity ∂f defined by a convex function f without a minimum. Suppose $f : \widetilde{M} \rightarrow \mathbb{R}$ is a convex function that does not attain its infimum. We want to describe a “center at infinity” to which the level sets $\{f = t\}$ converge as t approaches the infimum of f , generalizing the situation of horospheres $H(\xi, x)$ converging to the point at infinity ξ as x approaches ξ . For the convex function f , there may not be a single such distinguished point at infinity, but there is a set of points ∂f of diameter $\leq \pi/2$, defined in the following way.

Fix a basepoint $x_0 \in \widetilde{M}$ and replace f by $f - f(x_0)$, if necessary, so that $f(x_0) = 0$. For any $t \leq 0$, let x_t be the closest point to x_0 in the sublevel set $\{f \leq t\}$, and let $\xi_t \in \partial_\infty$ be the point at infinity obtained by extending the geodesic $[x_0, x_t]$ to a geodesic ray $[x_0, \xi_t)$. Let

$$(11) \quad \partial f := \{\text{limit points of sequences } \{\xi_{t_i}\} \text{ with } t_i \rightarrow \inf f\}.$$

4.3. The path x_t as a function of distance to x_0 . While x_t is a continuous path, it may look very different from a geodesic.¹³ However, we do have some control on how much it can “oscillate”.¹⁴ To express it, we first reparametrize x_t in terms of the distance to the basepoint $R = d(x_0, x_t)$. Since f is convex, for every $R \geq 0$ there is a unique $t(R) \leq 0$ so that the sublevel set $\{f \leq t(R)\}$ meets the sphere $S_{x_0}(R)$ at a single point $\lambda_R := x_{t(R)}$. Comparing with Euclidean triangles, we see how λ_R can vary with R .

Lemma 8. $d(\lambda_{R(1 \pm \delta)}, \lambda_R) \leq \sqrt{2\delta + \delta^2} R$.

¹³In fact, as far as we can tell, it may not be rectifiable.

¹⁴We will use this later in subsection 8.6.

Proof. The sublevel set $\{f \leq t(R)\}$ is convex, and λ_R is the point on it that is closest to x_0 . Since $t(R+\varepsilon R) \leq t(R)$, the point $\lambda_{R(1+\delta)}$ is also in the sublevel set $\{f \leq t(R)\}$. So, we can apply inequality (27) from subsection 4.1 with $C = \{f \leq t(R)\}$, $x = \lambda_R$ and $y = \lambda_{R(1+\delta)}$ to get $d(\lambda_R, \lambda_{R(1+\delta)}) \leq \sqrt{2\delta + \delta^2}R$. A similar argument with $C = \{f \leq t(R - \delta R)\}$, $x = \lambda_{R(1-\delta)}$ and $y = \lambda_R$ shows that $d(\lambda_{R(1-\delta)}, \lambda_R) \leq \sqrt{2\delta - \delta^2}R$. \square

4.4. Independence of basepoint. In the closed disk topology on $\widetilde{M} \cup \partial_\infty$, $\xi_{t_i} \rightarrow \xi$ if and only if $x_{t_i} \rightarrow \xi$. Convexity of f implies that $d(x_t, x'_t) \leq d(x_0, x'_0)$ for all $t < \min\{f(x_0), f(x'_0)\}$. So, any point $\xi \in \partial f$ is a limit $x_{t_i} \rightarrow \xi$ and hence also $x'_{t_i} \rightarrow \xi$. This implies that the set ∂f does not depend on the choice of the initial point x_0 .

4.5. Diameter $\leq \pi/2$. Inequality (10) implies $\angle_{x_0}(x_s, x_t) < \pi/2$. So, if $\xi, \eta \in \partial f$, then $\angle_{x_0}(\xi, \eta) \leq \pi/2$. Since this is true for every choice of basepoint x_0 , it follows that the Tits distance between ξ and η is $\leq \pi/2$. So,

$$(12) \quad \text{diameter}(\partial f) \leq \pi/2.$$

More generally, this argument shows that if g is another convex function such that every sublevel set of g contains a sublevel set of f , then

$$(13) \quad \text{diameter}(\partial f \cup \partial g) \leq \pi/2.$$

Write $g \prec f$ if for any $t > \inf g$ there is $s > \inf f$ so that $\{f \leq s\} \subset \{g \leq t\}$.¹⁵ Then \prec is a partial order on the set of convex functions without minimum. The above argument shows that for any chain $f_0 \prec \dots \prec f_k$ we have

$$(14) \quad \text{diameter}(\partial f_0 \cup \dots \cup \partial f_k) \leq \pi/2.$$

4.6. ρ -Invariance. For $\xi \in \partial_\infty$ and $x \in \widetilde{M}$, let $H(\xi, x)$ be the horosphere through x centered at ξ . It is shown in section 3.9 of [4] that if f is ρ -invariant, then $\rho H(\xi, x) = H(\xi, x)$ for every $\xi \in \partial f$.

Proof sketch. Let $A_t := \{f \leq t\}$ be a sublevel set and $f_t(x) := d(x, A_t)$ be the distance to it. One shows that if $\xi_{t_i} \rightarrow \xi$ for $t_i \rightarrow \inf f$, then the limit

$$(15) \quad h(x) := \lim_{i \rightarrow \infty} f_{t_i}(x) - f_{t_i}(x_0)$$

exists, is convex and has $|\nabla h| = 1$ everywhere. It follows from this that the integral curves of ∇h must be geodesics starting from ξ and that the level set $h = 0$ is the horosphere $H(\xi, x_0)$ centered at ξ . Since h is ρ -invariant, the horosphere is ρ -invariant, as well. \square

¹⁵In other words, $g \prec f$ if every sublevel set of g contains a sublevel set of f .

4.7. Invariant horospheres. Let $\gamma \in \Gamma$ be a parabolic. Then, the displacement function $d_\gamma(x) := d(x, \gamma x)$ is convex, does not attain its infimum, and is invariant under the group of elements commuting with γ ¹⁶

$$(16) \quad C_\gamma := \{\rho \in \Gamma \mid \rho\gamma = \gamma\rho\}.$$

So, C_γ preserves horospheres centered at points in ∂d_γ .

4.8. Fix sets. We will see in section 9 that these points at infinity will give divergent geodesic rays! To keep track of them we use the following notation. Let G be a group.

$$(17) \quad \text{Fix}(G) := \{\xi \in \partial_\infty \mid g\xi = \xi \text{ for all } g \in G\},$$

$$(18) \quad \text{Fix}^0(G) := \{\xi \in \partial_\infty \mid gH(\xi, x) = H(\xi, x) \text{ for all } g \in G\}.$$

So, $\text{Fix}(G)$ is the ordinary fix set, and $\text{Fix}^0(G)$ is the subset consisting of those points at infinity ξ such that any horosphere centered at ξ is preserved by G . In subsection 4.7 we saw that if G commutes with a parabolic element γ , then $\text{Fix}^0(G)$ is non-empty.

5. DISPLACEMENT FUNCTIONS FOR FREE ABELIAN GROUPS

In this section, $A < \pi_1 M$ will always be a free abelian group containing a parabolic element. We will construct a sort of displacement function f (see 5.10) for the group A that is a (possibly infinite) convex combination of displacement functions d_γ for elements $\gamma \in A$. This function will be convex, C_A -invariant, and without infimum, so we get from it a set of points at infinity ∂f . The assignment of this function to the group A depends on choices, but all possible choices give points at infinity within $\pi/2$ of each other. Moreover, if B is a subgroup of A containing a parabolic element, then we get a displacement function g corresponding to B and $\text{diameter}(\partial f \cup \partial g) \leq \pi/2$, and we get an analogous statement for chains of abelian subgroups (see Corollary 11 below). This will be used in subsection 10.3.

5.1. The infimum displacement $|\cdot|$. For any element $\gamma \in \Gamma$, denote by

$$(19) \quad |\gamma| := \inf_{x \in \widetilde{M}} d_\gamma(x)$$

the infimum of the displacement function of γ .

5.2. Commuting elements have intersecting sublevel sets. If γ and ρ commute, then the sublevel sets $\{d_\gamma \leq t\}$ and $\{d_\rho \leq s\}$ intersect. In fact, the intersection can be expressed as

$$(20) \quad \{d_\gamma \leq t\} \cap \{d_\rho \leq s\} = p_{\{d_\gamma \leq t\}}(\{d_\rho \leq s\}).$$

where $p_S : \widetilde{M} \rightarrow S$ is the closest point projection to a closed convex set S .

Remark. So, we can always find points where both γ and ρ are small.

¹⁶The group C_γ is called the *centralizer* of γ .

5.3. Infimum displacement of a finite convex combination. It is always true that $\inf(d_\gamma + d_\rho) \geq |\gamma| + |\rho|$. If γ and ρ commute, then it follows from the previous bullet that we can find points where both γ and ρ are simultaneously small, so we get $\inf(d_\gamma + d_\rho) = |\gamma| + |\rho|$. Similarly, if $\{\gamma_i\}_{i=1}^k$ are a finite collection of elements in an abelian group A , then for any non-negative ($a_i \geq 0$) linear combination,

$$(21) \quad \inf_{x \in \widetilde{M}} \left(\sum_{i=1}^k a_i d_{\gamma_i}(x) \right) = \sum_{i=1}^k a_i |\gamma_i|.$$

5.4. The infimum displacement $|\cdot|$ as a (semi)-norm. Lemma 6.6(2) of [4] shows that

$$(22) \quad |\gamma^n| = |n| |\gamma|.$$

For any pair of elements γ and ρ we have

$$(23) \quad d_{\gamma\rho}(x) := d(\gamma\rho x, x)$$

$$(24) \quad \leq d(\gamma\rho x, \gamma x) + d(\gamma x, x),$$

$$(25) \quad = d(\rho x, x) + d(\gamma x, x),$$

$$(26) \quad = d_\rho(x) + d_\gamma(x).$$

If γ and ρ commute, then we can find points where they are simultaneously small, so

$$(27) \quad |\gamma\rho| \leq |\gamma| + |\rho|.$$

This shows that $|\cdot|$ is a semi-norm on an abelian group A . If we quotient out by the subgroup

$$(28) \quad A^0 := \{\gamma \in A \mid |\gamma| = 0\}$$

of elements with vanishing infimum displacement, we get a norm $|\cdot|$ on A/A^0 . Note, in particular, that this implies A/A^0 is free abelian.¹⁷

5.5. Displacement function $d_{\mathcal{A}^0}$ for a basis of A^0 . Let $\mathcal{A}^0 = \{h_1, \dots, h_r\}$ be a basis for the group A^0 , and let

$$(29) \quad d_{\mathcal{A}^0} := d_{h_1} + \dots + d_{h_r}$$

be the sum of displacement functions of basis elements. If $\phi : A^0 \rightarrow A^0$ is a change of basis with matrix entries ϕ_{ij} , then

$$(30) \quad d_{\phi(h_1)}(x) + \dots + d_{\phi(h_r)}(x) \leq \sum_{i,j} |\phi_{ij}| d_{h_j}(x) \leq r \sup |\phi_{ij}| (d_{h_1}(x) + \dots + d_{h_r}(x)).$$

So there is a constant $C := C(\phi)$ for which

$$(31) \quad d_{\phi(\mathcal{A}^0)} < C \cdot d_{\mathcal{A}^0}.$$

¹⁷More explicitly, this follows from the homogeneity property (22): If $\gamma \in A$ is an element with $\gamma^n \in A^0$ then $|\gamma| = |\gamma^n|/n = 0$ implying that $\gamma \in A^0$. So A/A^0 has no finite order elements, and hence it is free abelian.

5.6. Elements of positive displacement. Pick a lift $\{\gamma_1, \dots, \gamma_k\}$ of a basis for the free abelian group A/A^0 and define the L^1 norm with respect to this basis via

$$(32) \quad |\gamma_1^{n_1} \dots \gamma_k^{n_k}|_1 := \sum_{i=1}^k |n_i| |\gamma_i|.$$

Then, for any non-identity element $\gamma := \gamma_1^{n_1} \dots \gamma_k^{n_k}$ we have the bound

$$(33) \quad \frac{d_\gamma(x)}{|\gamma|_1} \leq \sum \frac{|n_i| d_{\gamma_i}(x)}{|\gamma|_1} \leq \sum \frac{|n_i| d_{\gamma_i}(x)}{|n_i| |\gamma_i|} \leq \sum \frac{d_{\gamma_i}(x)}{|\gamma_i|},$$

which only depends on the displacements of the generators γ_i at x . Since all norms on a finite dimensional vector space are equivalent, we get a bound

$$(34) \quad \frac{d_\gamma(x)}{|\gamma|} \leq D_{\gamma_1, \dots, \gamma_k}(x),$$

where $D_{\gamma_1, \dots, \gamma_k}(x)$ only depends on displacements of the generators γ_i at x .

5.7. A k -dimensional p -series. It is well known that there is a number p , depending on k so that the infinite series

$$(35) \quad \sum_{1 \neq \gamma \in \mathbb{Z}^k} \frac{1}{|\gamma|^{p-1}}$$

converges. We will fix a p that works for all the free abelian groups in $\pi_1 M$.

5.8. Displacement series d_{A^+} for elements of positive displacement. Now, let $A^+ := \langle \gamma_1, \dots, \gamma_k \rangle$ be a lift of the group A/A^0 and define the displacement series

$$(36) \quad d_{A^+}(x) := \sum_{1 \neq \gamma \in A^+} \frac{d_\gamma(x)}{|\gamma|^p}.$$

This series converges for every x because it is bounded by

$$(37) \quad d_{A^+}(x) \leq D_{\gamma_1, \dots, \gamma_k}(x) \cdot \sum_{1 \neq \gamma \in A^+} \frac{1}{|\gamma|^{p-1}}.$$

It follows from (37) and (21) that

$$(38) \quad \inf_{x \in \widetilde{M}} d_{A^+}(x) = \sum_{1 \neq \gamma \in A^+} \frac{1}{|\gamma|^{p-1}}.$$

Remark. The point is that we use (21) to compute the infimum of the partial sum and (37) to show that the infimum of the tail goes to zero.

5.9. Changing the lift. Pick different lifts $\phi(\gamma_i) = \gamma_i g_i$ for $g_i \in A^0$, and let $\phi(A^+) := \langle \gamma_1 g_1, \dots, \gamma_k g_k \rangle$. Then, for $\gamma = \gamma_1^{n_1} \dots \gamma_k^{n_k}$ we have $\phi(\gamma) = (\gamma_1 g_1)^{n_1} \dots (\gamma_k g_k)^{n_k}$ and compute

$$(39) \quad |d_\gamma(x) - d_{\phi(\gamma)}(x)| \leq \sum_i |n_i| d_{g_i}(x) \leq \sup_i d_{g_i}(x) \cdot \sum |n_i|.$$

Consequently

$$(40) \quad |d_{A^+}(x) - d_{\phi(A^+)}(x)| \leq \sup_i d_{g_i}(x) \cdot \sum_{0 \neq (n_1, \dots, n_k) \in \mathbb{Z}^k} \frac{|n_1| + \dots + |n_k|}{|\gamma_1^{n_1} \dots \gamma_k^{n_k}|^p}.$$

Remark. Using the L^1 norm, one sees that the series on the right converges.

Express the elements g_i in terms of the basis \mathcal{A}^0 , i.e. $g_i = h_1^{\psi_{i1}} \dots h_r^{\psi_{ir}}$. Then, there is a constant $D := D(\psi_{ij})$ so that

$$(41) \quad |d_{A^+}(x) - d_{\phi(A^+)}(x)| \leq D \cdot d_{\mathcal{A}^0}(x).$$

5.10. The combined displacement function. Recall that A is a free abelian group containing a parabolic element and A^0 is the subgroup of isometries with vanishing inf displacement. We pick a basis \mathcal{A}^0 for A^0 and a lift A^+ of A/A^0 . Now, look at the function

$$(42) \quad f = f_{\mathcal{A}^0, A^+} := d_{\mathcal{A}^0} + d_{A^+}.$$

This function is convex and C_A -invariant. Since A has a parabolic, f does not attain its infimum.¹⁸ So, we get a set of points at infinity ∂f such that

- the set ∂f has diameter $\leq \pi/2$, and
- horospheres centered at $\xi \in \partial f$ are preserved by the centralizer C_A .

5.11. Splicing sublevel sets of convex functions. The function $f = f_{\mathcal{A}^0, A^+}$ depends on the choice of basis for \mathcal{A}^0 and lift A^+ , so the set of limit points ∂f also depends on these choices. Next, we will show that for a different basis $\phi(\mathcal{A}^0)$, a different lift $\phi(A^+)$, and a corresponding function $g := f_{\phi(\mathcal{A}^0), \phi(A^+)}$, the union $(\partial f \cup \partial g)$ has diameter $\leq \pi/2$.

Remark. The infimum of $f_{\mathcal{A}^0, A^+}$ does not depend on either the choice of basis \mathcal{A}^0 or the choice of lift A^+ , and is equal to $\inf d_{A^+}$.

Lemma 9. *Every sublevel set of g contains a sublevel set of f .*

Proof. Suppose that

$$(43) \quad f(x) = d_{\mathcal{A}^0}(x) + d_{A^+}(x) < \varepsilon + \inf d_{A^+}.$$

¹⁸The function f is of the form $d_{h_1} + \dots + d_{h_r} + \sum_{\gamma \in A^+} a_\gamma d_\gamma$ for some positive constants a_γ . It follows from (21) and (37) that its infimum is $\sum a_\gamma |\gamma|$. So, if f attains its infimum, then so do all the functions d_{h_i} and $\{d_\gamma\}_{\gamma \in A^+}$. This can only happen if $A^0 = 0$ and $A = A^+$ consists of semisimple elements. This contradicts the assumption that A contains a parabolic. So f does not attain its infimum.

Then

$$(44) \quad d_{A^+}(x) < \varepsilon + \inf d_{A^+},$$

$$(45) \quad d_{\mathcal{A}^0}(x) < \varepsilon.$$

Together with (31) the later of these implies that

$$(46) \quad d_{\phi(\mathcal{A}^0)}(x) < C\varepsilon,$$

while together with (41) it implies that

$$(47) \quad |d_{\phi(A^+)}(x) - d_{A^+}(x)| < D\varepsilon.$$

Consequently,

$$(48) \quad g(x) = d_{\phi(\mathcal{A}^0)}(x) + d_{\phi(A^+)}(x)$$

$$(49) \quad = d_{\phi(\mathcal{A}^0)}(x) + (d_{\phi(A^+)}(x) - d_{A^+}(x)) + d_{A^+}(x)$$

$$(50) \quad \leq C\varepsilon + D\varepsilon + \varepsilon + \inf d_{A^+}$$

$$(51) \quad = (C + D + 1)\varepsilon + \inf d_{A^+}.$$

Picking ε sufficiently small, we get that every sublevel set of g contains a sublevel set of f . \square

In the notation of 4.5 we have $g \prec f$ and, by symmetry, also $f \prec g$.

Remark. The diameter bound (13) implies

$$(52) \quad \text{diameter}(\partial f \cup \partial g) \leq \pi/2.$$

So, the set of points $\cup \partial f_{\mathcal{A}^0, A^+}$ obtained from all possible choices of basis \mathcal{A}^0 and lifts A^+ also has diameter $\leq \pi/2$.

5.12. Comparing displacement functions for a free abelian group A and a subgroup B . Suppose B is a subgroup of A , and suppose B contains a parabolic. Let \mathcal{B}^0 be a basis for B^0 and pick a lift B^+ of $B/B^0 < A/A^0$ extending to a lift A^+ . Expressing elements of \mathcal{B}^0 in the basis \mathcal{A}^0 , we get a constant C so that

$$(53) \quad d_{\mathcal{B}^0} \leq C \cdot d_{\mathcal{A}^0}.$$

Moreover, we also have

$$(54) \quad d_{B^+}(x) - \inf d_{B^+} \leq d_{A^+}(x) - \inf d_{A^+}$$

because

$$d_{A^+}(x) - \inf d_{A^+} = d_{B^+}(x) - \inf d_{B^+} + \left(\sum_{\gamma \in A^+ \setminus B^+} \frac{d_\gamma(x) - |\gamma|}{|\gamma|^p} \right).$$

Lemma 10. *Every sublevel set of $f_{\mathcal{B}^0, B^+}$ contains a sublevel set of $f_{\mathcal{A}^0, A^+}$.*

Proof. Starting as in Lemma 9 with $f_{\mathcal{A}^0, A^+}(x) < \varepsilon + \inf d_{A^+}$, we get

$$(55) \quad d_{A^+}(x) < \varepsilon + \inf d_{A^+}.$$

$$(56) \quad d_{\mathcal{A}^0}(x) < \varepsilon.$$

Together with (53), the second of these gives

$$(57) \quad d_{\mathcal{B}^0}(x) < C\varepsilon.$$

Rewriting the first as $d_{A^+}(x) - \inf d_{A^+} < \varepsilon$ and using (54) we get

$$(58) \quad d_{B^+}(x) - \inf d_{B^+} \leq d_{A^+}(x) - \inf d_{A^+} < \varepsilon,$$

and therefore

$$(59) \quad d_{B^+}(x) < \varepsilon + \inf d_{B^+}.$$

Consequently,

$$(60) \quad f_{\mathcal{B}^0, B^+}(x) = d_{\mathcal{B}^0}(x) + d_{B^+}(x),$$

$$(61) \quad < C\varepsilon + \varepsilon + \inf d_{B^+}.$$

As in Lemma 9, this shows that every sublevel set of $f_{\mathcal{B}^0, B^+}$ contains a sublevel set of $f_{\mathcal{A}^0, A^+}$. \square

Together, these two lemmas lead to the main result of this section.

Corollary 11. *Suppose $A_0 < \dots < A_k$ is a chain of free abelian groups and A_0 contains a parabolic element. Let $f_i = d_{\mathcal{A}_i^0} + d_{A_i^+}$. Then*

$$(62) \quad \text{diameter}(\partial f_0 \cup \dots \cup \partial f_k) \leq \pi/2.$$

Proof. Lemma 10 says that

$$(63) \quad f_{\mathcal{B}^0, B^+} \prec f_{\mathcal{A}^0, A^+}$$

if A^+ is a lift extending B^+ . Applying Lemma 9 we see that (63) is true for any lift A^+ . So, we get $f_0 \prec \dots \prec f_k$ and (14) gives the diameter bound. \square

Remark. If $B < A$, $\gamma_1, \dots, \gamma_r$ is a basis for B and ρ_1, \dots, ρ_k is a basis for A , then expressing the γ_i in terms of the ρ_i gives an inequality

$$g = d_{\gamma_1} + \dots + d_{\gamma_r} \leq C \cdot (d_{\rho_1} + \dots + d_{\rho_k}) = C \cdot f.$$

If one could deduce from this inequality that every sublevel set of g contains a sublevel set of f , i.e. that $g \prec f$, then one could use $d_{\rho_1} + \dots + d_{\rho_k}$ as the displacement function for A , and this section would be considerably simpler. However, we do not know how to do this (except in the simple case when all the elements ρ_i have zero inf displacement). This is why, instead, we build the more complicated displacement functions $f = d_{\mathcal{A}^0} + d_{A^+}$ and $g = d_{\mathcal{B}^0} + d_{B^+}$ which involve all the elements of positive inf displacement and for which we can show $g \prec f$.

6. BUSEMANN SIMPLICES

Everything in the first three subsections below is in chapter 3 of [4].

6.1. Horofunctions, Busemann functions and points at infinity. Sending a point $x \in \widetilde{M}$ to the distance function to that point $d_x(y) := d(x, y)$ embeds \widetilde{M} inside the space of continuous functions on \widetilde{M}

$$(64) \quad \iota : \widetilde{M} \hookrightarrow C(\widetilde{M}),$$

$$(65) \quad x \mapsto d_x.$$

The image of this map is not closed, because a sequence of distance functions $\{d_{x_i}\}$ can converge even if the sequence of points $\{x_i\}$ does not. A function $h \in \overline{\iota(\widetilde{M})} \setminus \iota(\widetilde{M})$ which is a limit of distance functions but is not a distance function itself is called a *horofunction*. A horofunction h is a limit of convex functions with unit gradient, so it is also a convex function with unit gradient. From this, one can show that h is a *Busemann function*. That is, there is a geodesic ray $r : [0, \infty) \rightarrow \widetilde{M}$ so that $h(x) = \lim_{s \rightarrow \infty} (d(y, r(s)) - s)$.

6.2. Endpoints. The endpoint $r(\infty)$ of the geodesic ray r can be recovered from h in various ways. For instance, it is the endpoint of any geodesic ray $\exp_{x_0}(-t\nabla h)$ starting at any basepoint $x_0 \in \widetilde{M}$ and going in the direction of $-\nabla h$. Alternatively, since h is a Busemann function it is not hard to see that $\partial h = \{r(\infty)\}$. We will denote this point simply as \bar{h} .

6.3. Sphere at infinity via horofunctions. Let *Horo* be the space of all horofunctions on \widetilde{M} . Then

$$(66) \quad \text{Horo} \rightarrow \partial_\infty \times \mathbb{R}$$

$$(67) \quad h \mapsto (\bar{h}, h(x_0))$$

is a homeomorphism, describing the boundary at infinity via horofunctions.

6.4. Busemann simplices. If h_0, \dots, h_k is a collection of horofunctions so that the set of points at infinity $\{\bar{h}_0, \dots, \bar{h}_k\}$ has diameter $\leq \pi/2$, then the convex combination $t_0 h_0 + \dots + t_k h_k$ is a convex function that does not attain its infimum. So, there is a set of points at infinity $\partial(t_0 h_0 + \dots + t_k h_k)$ corresponding to it.¹⁹ A *Busemann simplex* (with vertices $\{\bar{h}_0, \dots, \bar{h}_k\}$) is a singular simplex σ so that $\sigma(t_0, \dots, t_k) \in \partial(t_0 h_0 + \dots + t_k h_k)$ for every $t \in \Delta$. We construct such simplices in the next section, and also show that the simplices we construct are “not too bad” (Lipschitz).

6.5. G -invariant horofunctions. If the horofunctions h_0, \dots, h_k are G -invariant, then so is the convex combination $t_0 h_0 + \dots + t_k h_k$, so all the points at infinity $\partial(t_0 h_0 + \dots + t_k h_k)$ are contained in $\text{Fix}^0(G)$. So, a Busemann simplex σ with vertices in $\text{Fix}^0(G)$ is entirely contained in $\text{Fix}^0(G)$.

¹⁹For non-trivial combinations this set could contain more than one point.

7. CONSTRUCTION OF BUSEMANN SIMPLICES

In this section, we will first construct Busemann simplices in Lemma 12.

Lemma 12. *Let h_0, h_1, \dots, h_k be Busemann functions so that the set of points at infinity $\{\bar{h}_0, \bar{h}_1, \dots, \bar{h}_k\}$ has diameter $\leq \pi/2$. Then there is a Lipschitz map $\sigma: \Delta^k \rightarrow \partial_\infty \widetilde{M}$ such that $\sigma(t_0, t_1, \dots, t_k) \in \partial(t_0 h_0 + t_1 h_1 + \dots + t_k h_k)$ for every $t = (t_0, t_1, \dots, t_k) \in \Delta^k$. That is, σ is a Busemann simplex.*

Then we will deduce as a corollary of the proof of Lemma 12 how to construct *Busemann polyhedra*, which is what we really need in the proof of the main theorem. Note that Corollary 13 does not follow from the statement of Lemma 12 since it is not clear if Busemann simplices are unique.

Corollary 13. *Suppose that P is a finite polyhedron and $\rho^{(0)}$ is a map on the set of vertices of P to $\partial_\infty \widetilde{M}$ such that for each simplex Δ^k of P , the image of the vertices of Δ^k under $\rho^{(0)}$ has diameter $\leq \pi/2$. Then $\rho^{(0)}$ extends to a Lipschitz map $\rho: P \rightarrow \partial_\infty \widetilde{M}$ such that for each simplex Δ^k of P the restriction of ρ to Δ^k is a Busemann simplex.*

Before we prove Lemma 12, we set up notations and make observations. For each $t = (t_0, t_1, \dots, t_k) \in \Delta^k$, let

$$f_t := \sum_{i=0}^k t_i h_i = (1 - t_1 - \dots - t_k) h_0 + t_1 h_1 + \dots + t_k h_k.$$

We have a few observations.

- a) The function f_t is convex and it does not have a minimum because the gradient

$$\nabla f_t = \sum_{i=0}^k t_i \nabla h_i$$

is never zero since it is a positive combination of nonzero vectors $\nabla h_0, \nabla h_1, \dots, \nabla h_k$ that are mutually at an angle at most $\pi/2$. Therefore, ∂f_t is nonempty.

- b) The length of the gradient ∇f_t is bounded by

$$\frac{1}{\sqrt{k+1}} \leq |\nabla f_t| \leq 1$$

since $\nabla h_0, \nabla h_1, \dots, \nabla h_k$ have unit length and $\nabla h_0, \nabla h_1, \dots, \nabla h_k$ are mutually at an angle at most $\pi/2$.

Proof of Lemma 12. We first prove the lemma for the case of a path, i.e. for $t \in \Delta^1$. So

$$f_t = t_0 h_0 + t_1 h_1 = (1 - t_1) h_0 + t_1 h_1.$$

After that we will discuss how this generalize to higher dimensional simplices. For simplicity of notation, in this case we will write $\Delta^1 = [0, 1]$, i.e. we equate t with t_1 , and

$$f_t = (1 - t) h_0 + t h_1.$$

We define the path σ as follows.

- (i) For each dyadic rational $s \in [0, 1] = \Delta^1$, we pick an *appropriate* point in ∂f_s and define $\sigma(s)$ to be that point.
- (ii) Then we extend σ to for all $t \in [0, 1]$. To make this extension work we need to make sure that a Cauchy sequence (s_i) of dyadic rational numbers in $[0, 1]$ gives a Cauchy/convergent sequence $(\sigma(s_i))$ in $(\partial_\infty \widetilde{M}, \angle)$. This is the reason why the choice of $\sigma(t)$ above has to be *appropriate*.

Defining σ on dyadic rational points. Fix a base point $x \in \widetilde{M}$ and $R > 0$, and take the sphere $S_x(R)$ centered at x with radius R . For each $t \in [0, 1]$, there is a unique sublevel set $f_t \leq a_{t,R}$ that touches $S_x(R)$ at a (unique) point which we call $\sigma_R(t)$. Note that $\sigma_R(t)$ is the closest point projection of x onto the sublevel set $f_t \leq a_{t,R}$. We define σ on the set of dyadic rational numbers in $[0, 1]$ as follows.

- Note that

$$\bar{h}_0 = \lim_{R \rightarrow \infty} \sigma_R(0) \quad \text{and} \quad \bar{h}_1 = \lim_{R \rightarrow \infty} \sigma_R(1).$$

Let $\sigma(0) = \bar{h}_0$ and $\sigma(1) = \bar{h}_1$.

- There is a sequence $R_n \rightarrow \infty$ such that $(\sigma_{R_n}(1/2))$ has only one limit point. Let

$$\sigma(1/2) = \lim_{n \rightarrow \infty} \sigma_{R_n}(1/2).$$

- There is a subsequence (R_{n_k}) of (R_n) such that each of $(\sigma_{R_{n_k}}(1/4))$ and $(\sigma_{R_{n_k}}(3/4))$ has only one limit point. Let

$$\sigma(1/4) = \lim_{n_k \rightarrow \infty} \sigma_{R_{n_k}}(1/4) \quad \text{and} \quad \sigma(3/4) = \lim_{n_k \rightarrow \infty} \sigma_{R_{n_k}}(3/4).$$

- There is a subsequence $(R_{n_{k_l}})$ of (R_{n_k}) such that each of $(\sigma_{R_{n_{k_l}}}(p/8))$, for $p = 1, 3, 5, 7$, has only one limit point. Let

$$\sigma(p/8) = \lim_{n_{k_l} \rightarrow \infty} \sigma_{R_{n_{k_l}}}(p/8), \quad \text{for } p = 1, 3, 5, 7.$$

- And so on.

In order to show that σ is continuous we first take the metric \angle_x on $S_x(R)$, where R can be ∞ , to be given by the angle at x . Then we show that the map σ_R , for each $0 < R < \infty$, is Lipschitz with the same Lipschitz constant. This implies that σ , as a function defined on the set of dyadic rational points of $[0, 1]$, is also Lipschitz. Therefore, the image of a Cauchy sequence under σ is a Cauchy sequence. Hence, we can extend σ to a continuous (in fact, Lipschitz) map on $[0, 1]$. So, we need to prove the following lemma.

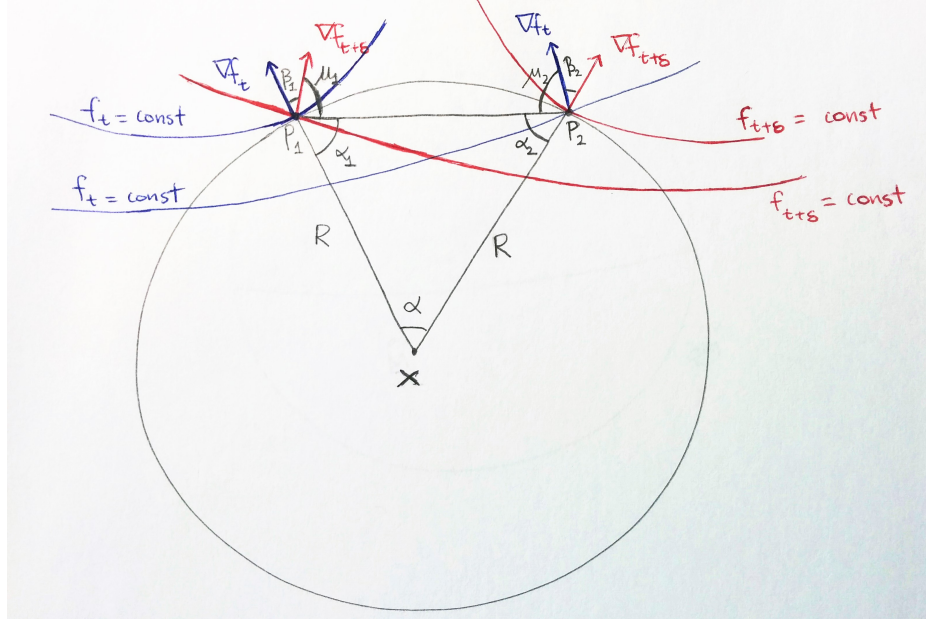


FIGURE 7.

Lemma 14. For each $R > 0$, the map

$$\sigma_R: \Delta^k \rightarrow (S_x(R), \angle_x)$$

is Lipschitz. In particular, for $t \in \Delta^k$ and for $\delta = (\delta_0, \delta_1, \dots, \delta_k)$ with $|\delta|_1$ small and such that $(t + \delta) \in \Delta^k$, we have

$$\angle_x(\sigma_R(t), \sigma_R(t + \delta)) \leq 6k|\delta|_1,$$

where $|\delta|_1 = |\delta_0| + |\delta_1| + \dots + |\delta_k|$ is the L^1 metric.

Proof. The main idea of this proof is in Figure 7, which the reader is encouraged to look at if they try to follow what is written next.

Fix $R > 0$ and δ . Let $p_1 = \sigma_R(t)$ and $p_2 = \sigma_R(t + \delta)$. Let $\alpha = \angle_x(p_1, p_2)$. Our goal is to bound α in terms of δ . For each $i = 1, 2$,

- Let β_i be the angle at p_i between ∇f_t and $\nabla f_{t+\delta}$,
- Let $\alpha_i = \angle_{p_i}(x, p_{i+1})$, where addition in i is taken mod 2,
- Let μ_1 be the angle at p_1 between the $\nabla f_{t+\delta}$ and the tangent to the geodesic from p_1 to p_2 , and
- Let μ_2 be the angle at p_2 between the ∇f_t and the tangent to the geodesic from p_2 to p_1 .

Then

$$\alpha_1 + \mu_1 + \beta_1 \geq \pi$$

since at p_1 the vector ∇f_t is parallel to the tangent vector to the geodesic from x to p_1 because they are both orthogonal to the level set $f_t = f_t(p_1)$.

Similarly,

$$\alpha_2 + \mu_2 + \beta_2 \geq \pi.$$

Therefore,

$$\beta_1 + \beta_2 \geq (\pi - \alpha_1 - \alpha_2) + (\pi - \mu_1 - \mu_2).$$

Now, α , α_1 and α_2 are the three angles of the triangle $\triangle xp_1p_2$. Thus,

$$\alpha + \alpha_1 + \alpha_2 \leq \pi, \quad \text{so} \quad \alpha \leq \pi - \alpha_1 - \alpha_2.$$

Hence,

$$\beta_1 + \beta_2 \geq \alpha + (\pi - \mu_1 - \mu_2).$$

Next, we show that $\mu_1 \leq \pi/2$ and $\mu_2 \leq \pi/2$, so that $\beta_1 + \beta_2 \geq \alpha$, which we can use to bound α in terms of δ .

To see that $\mu_1 \leq \pi/2$, observe that p_1 does not belong to the sublevel set $f_{t+\delta} \leq f_{t+\delta}(p_2)$ since the sphere $S_x(R)$ lies on the other side of the level set $f_{t+\delta} = f_{t+\delta}(p_2)$. Therefore, $f_{t+\delta}(p_1) > f_{t+\delta}(p_2)$, so p_2 is contained in the sublevel set $f_{t+\delta} \leq f_{t+\delta}(p_1)$. Since at p_1 the vector $\nabla f_{t+\delta}$ is orthogonal to the level set $f_{t+\delta} = f_{t+\delta}(p_1)$ it follows that $\mu_1 \leq \pi/2$. Similarly, we see that $\mu_2 \leq \pi/2$ and we obtain that

$$\alpha \leq \beta_1 + \beta_2.$$

Next, we bound β_i , for $i = 1, 2$, in terms of δ .

$$f_{t+\delta} = f_t + \sum_{i=1}^k \delta_i (h_i - h_0).$$

So

$$\nabla f_{t+\delta} = \nabla f_t + \sum_{i=1}^k \delta_i (\nabla h_i - \nabla h_0).$$

Take the orthogonal projection in the tangent space $T_{p_1} \widetilde{M}$ in the direction of ∇f_t . We have

$$(\nabla f_{t+\delta})^\perp = (\nabla f_t)^\perp + \sum_{i=1}^k \delta_i (\nabla h_i - \nabla h_0)^\perp,$$

where $(\nabla f_{t+\delta})^\perp$ is the component of $\nabla f_{t+\delta}$ that is orthogonal to ∇f_t , etc. Since $(\nabla f_t)^\perp = 0$, we have

$$(\nabla f_{t+\delta})^\perp = \sum_{i=1}^k \delta_i (\nabla h_i - \nabla h_0)^\perp.$$

Thus,

$$|(\nabla f_{t+\delta})^\perp| \leq \sum_{i=1}^k |\delta_i| (|\nabla h_i| + |\nabla h_0|) \leq 2|\delta|_1.$$

Now

$$|(\nabla f_{t+\delta})^\perp| = |\nabla f_{t+\delta}| \sin \beta_1 \geq \frac{1}{\sqrt{k+1}} \sin \beta_1$$

since $|\nabla f_{t+\delta}| \geq 1/\sqrt{k+1}$ as we observed earlier. Therefore,

$$\sin \beta_1 \leq 2\sqrt{k+1}|\delta|_1.$$

When β_1 is small, $\sin \beta_1 \approx \beta_1$, so we can replace $\sin \beta_1 \leq 2\sqrt{k+1}|\delta|_1$ by $\beta_1 \leq 3k|\delta|_1$.

Similarly, we can show that $\beta_2 \leq 3k|\delta|_1$. Hence,

$$\alpha \leq \beta_1 + \beta_2 \leq 6k|\delta|_1,$$

and the proof is complete. \square

It follows that σ is Lipschitz since it is the limit of Lipschitz maps with the same Lipschitz constant.

Now we discuss how to generalize the above proof to the general case with more than two Busemann functions h_0, h_1, \dots, h_k . This extension is straightforward. Instead of using dyadic rational numbers, we will use “barycentric rational points” in Δ^k . A *barycentric rational point* in Δ^k is a vertex in the m -th barycentric subdivision of Δ^k for some $m = 1, 2, 3, \dots$. It is not hard to see that the set of barycentric rational points of Δ^k is dense in Δ^k .

In the exact same way we define σ for a path, we define σ at the vertices of Δ^k , and then on the vertices of the first barycentric subdivision of Δ^k , and then on the vertices of the second barycentric subdivision of Δ^k , etc. By Lemma 14, we can extend σ to the whole Δ^k and get a Lipschitz map

$$\sigma: \Delta^k \rightarrow (\partial_\infty \widetilde{M}, \angle_x).$$

What is left to show is that $\sigma(t_0, t_1, \dots, t_k) \in \partial(t_0 h_0 + t_1 h_1 + \dots + t_k h_k)$ for every $t = (t_0, t_1, \dots, t_k) \in \Delta^k$. This is true when t is a barycentric rational point, so we need to check that it is also for other $t \in \Delta^k$. First we note the following immediate corollary of Lemma 14.

Corollary 15. *Let t and δ be as in Lemma 14. Then $\partial f_{t+\delta}$ is contained in $(6k|\delta|_1)$ -neighborhood of ∂f_t in the \angle_x metric.*

Let $t \in \Delta^k$ be a barycentric irrational point. Then there is a sequence of barycentric rational points t^j in Δ^k that approximates t . Assume that $|t^j - t|_1 \leq 1/j$. By Corollary 15, for each j , there is $\xi_j \in \partial f_t$ such that

$$\angle_x(\sigma(t^j), \xi_j) \leq \frac{6k}{j} \quad \text{for all } j = 1, 2, 3, \dots$$

Since $\sigma(t^j) \rightarrow \sigma(t)$, it follows that $\xi_j \rightarrow \sigma(t)$. Now, the set ∂f_t is closed since it is a limit point set. So $\sigma(t) \in \partial f_t$. Thus, $\sigma(t_0, t_1, \dots, t_k) \in \partial(t_0 h_0 + t_1 h_1 + \dots + t_k h_k)$ for every $t = (t_0, t_1, \dots, t_k) \in \Delta^k$. This completes the proof. \square

To see that Corollary 13 is true, we define ρ on P the same way as in the proof of Lemma 12. That is, we define ρ on barycentric rational points of P by defining it on the vertices of P , then of the first barycentric subdivision

of P , etc. It follows for the same reason as in the proof of Lemma 12 that ρ is Lipschitz and therefore we can extend it to all of P . It follows that the restriction of ρ to each simplex in P is a Busemann simplex. This proves Corollary 13.

8. A DIMENSION BOUND

8.1. Non-degenerate simplices. Let $\Delta^k := \{(t_0, \dots, t_k) \in \mathbb{R}_{\geq 0}^{k+1} \mid t_0 + \dots + t_k = 1\}$ be the standard k -simplex. Recall that a singular k -simplex in X is a continuous map $\lambda : \Delta^k \rightarrow X$. It is *non-degenerate* if the image $\lambda(\Delta)$ is not contained in the image of the boundary $\lambda(\partial\Delta)$. Any point $x \in \lambda(\Delta) \setminus \lambda(\partial\Delta)$ is called a *non-degenerate point*. The meaning of non-degeneracy for the finite approximations σ_R is, partly, explained by the following lemma.

Lemma 16. *If $x \in \sigma_R(\Delta)$ is non-degenerate, then the gradient vectors $\{\nabla h_0, \dots, \nabla h_k\}$ are linearly independent at x .*

Proof. Let $f_t = t_0 h_0 + \dots + t_k h_k$. At $x := \sigma_R(t) \in S_{x_0}(R)$, the gradient

$$\nabla f_t = t_0 \nabla h_0 + \dots + t_k \nabla h_k$$

is perpendicular to the sphere $S_{x_0}(R)$. Suppose the $\{\nabla h_i\}$ are linearly dependent at x . Reordering the h_i , we can write this linear dependence as

$$(68) \quad s_0 \nabla h_0 + \dots + s_j \nabla h_j = s_{j+1} \nabla h_{j+1} + \dots + s_l \nabla h_l,$$

for some $0 \leq j < l \leq k$ and *positive* constants $\{s_i\}_{i=0}^l$. Rescale this relation by dividing by the constant $m := \max_i (s_i/t_i)$ to get

$$(69) \quad \frac{s_0}{m} \nabla h_0 + \dots + \frac{s_j}{m} \nabla h_j = \frac{s_{j+1}}{m} \nabla h_{j+1} + \dots + \frac{s_l}{m} \nabla h_l.$$

In this relation, all the coefficients satisfy $(s_i/m) \leq t_i$ and for (at least) one of the coefficients, this inequality is an equality. We may assume this happens for the zero'th coefficient, i.e. $s_0/m = t_0$. Now, adding the non-negative linear combination

$$\left(t_1 - \frac{s_1}{m}\right) \nabla h_1 + \dots + \left(t_j - \frac{s_j}{m}\right) \nabla h_j + t_{j+1} \nabla h_{j+1} + \dots + t_k \nabla h_k$$

to both sides of equation (69), we get

$$(70) \quad \nabla f_t = \nabla(a_1 h_1 + \dots + a_k h_k)$$

for some non-negative constants $\{a_i\}_{i=1}^k$. Setting $t' := (0, \frac{a_1}{|a|_1}, \dots, \frac{a_k}{|a|_1}) \in \partial\Delta$, equation (70) can be rewritten as $\nabla f_t = |a|_1 \nabla f_{t'}$ at the point x , which implies that $x = \sigma_R(t')$. So, x is a degenerate point. \square

Next, we look at all the preimages of non-degenerate points

$$(71) \quad U := \{(R, t) \in \mathbb{R}^+ \times \Delta \mid \sigma_R(t) \text{ is non-degenerate}\},$$

This set is open because linear independence of $\{\nabla h_i\}$ is an open condition.

Corollary 17. *The map $U \rightarrow \widetilde{M}$, $(R, t) \mapsto \sigma_R(t)$ is injective.*

Proof. Note that $\sigma_R(t) = x = \sigma_R(t')$ gives the linear relation

$$t_0 \nabla h_0 + \cdots + t_k \nabla h_k = c(t'_0 \nabla h_0 + \cdots + t'_k \nabla h_k)$$

at the point x . If x is non-degenerate, the previous lemma implies this relation is trivial. So, we must have $t = t'$. This proves the corollary. \square

8.2. Dimension bound. For concreteness, suppose in this section that $G = \mathbb{Z}^r$ is a free abelian group of rank r .²⁰ Our goal in this section is the following.

Theorem 18. *If $\text{Fix}^0(\mathbb{Z}^r)$ has a non-degenerate Busemann k -simplex then*

$$(72) \quad \dim \widetilde{M} \geq r + 1 + k.$$

The parameter space for intersections of horospheres. Pick representatives h_i for the vertices of the Busemann simplex $\sigma : \Delta^k \rightarrow \text{Fix}^0(G)$, so that $h_i(x_0) = 0$, and look at the “horospherical coordinates” map

$$(73) \quad F : \widetilde{M} \rightarrow \mathbb{R}^{k+1},$$

$$(74) \quad x \mapsto (h_0(x), \dots, h_k(x)).$$

The map F is (obviously) \mathbb{Z}^r -invariant, and its image $F(\widetilde{M})$ in \mathbb{R}^{k+1} is the parameter space for all possible intersections of horospheres $\cap_{i=0}^k \{h_i = b_i\}$.

Choice of metric on \mathbb{R}^{k+1} . Unless we specify otherwise, the distance on \mathbb{R}^{k+1} will always be the one given by the sup norm

$$d(a, b) := |a - b|_\infty.$$

This is convenient because with this metric F is a contraction.

The philosophy²¹ behind the argument. Our goal is to show that the dimension of \widetilde{M} is $\geq r + k + 1$. A natural way to try and do this is by showing that each intersection of horospheres $F^{-1}(b)$ has dimension $\geq r$ and that there is a $(k+1)$ -dimensional family of such intersections of horospheres. The map F is \mathbb{Z}^r -invariant, and the group \mathbb{Z}^r acts on each intersection of horospheres $F^{-1}(b)$, but this by itself is not enough to bound the dimension of $F^{-1}(b)$ from below by r . In fact $F^{-1}(b)$ could be discrete. However, if we can show that there is a \mathbb{Z}^r -equivariant map $\mathbb{R}^r \rightarrow F^{-1}(b)$ then this would imply that $\dim F^{-1}(b) \geq r$.²² It turns out that we can't quite do this either, but we can do it “up to a uniformly bounded error L ” for a family of intersections of horospheres W that is determined by the simplex σ . If the simplex σ is non-degenerate, then this family is $(k+1)$ -dimensional. In

²⁰All the arguments in this section apply to any discrete group G with finite classifying space BG . In the argument, we replace r by maximal dimension of a non-trivial homology class in BG and \mathbb{R}^r by the universal cover EG of BG .

²¹In the mathematical sense.

²²Because taking \mathbb{Z}^r -quotients gives a map $\mathbb{T}^r \rightarrow F^{-1}(b)/\mathbb{Z}^r$ whose composition with the “classifying map of the \mathbb{Z}^r -cover” $F^{-1}(b)/\mathbb{Z}^r \rightarrow \mathbb{T}^r$ is a homotopy equivalence.

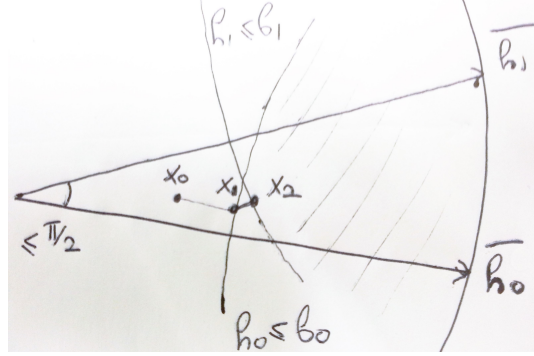


FIGURE 8.

fact, it is “fat”, which means that it contains balls of arbitrarily large radius. This fatness compensates for the error L , and leads to the dimension bound.

The setup. Let $\sigma_{>0}(\Delta)$ be the union $\cup_{R>0} \sigma_R(\Delta)$ and $W := F(\sigma_{>0}(\Delta))$ its image under the horospherical coordinate map. For $b = (b_0, \dots, b_k) \in \mathbb{R}^{k+1}$ and $x \in \widetilde{M}$, let $p(b, x)$ be closest point projection of x to the intersection of horoballs $\cap_i \{h_i \leq b_i\}$, and note that it is \mathbb{Z}^r -equivariant in the second factor. Extend the orbit $\mathbb{Z}^r x_0$ to a \mathbb{Z}^r -equivariant map $f : \mathbb{R}^r \rightarrow \widetilde{M}$, $f(0) = x_0$ and project to the intersection of horoballs to get a \mathbb{Z}^r -equivariant map

$$(75) \quad f_b : \mathbb{R}^r \rightarrow \widetilde{M},$$

$$(76) \quad f_b(z) := p(b, f(z)).$$

Bounding projections in horospherical coordinates. We will need the following simple lemma, bounding distance between x and its closest point projection $p(b, x)$ to an intersection of horoballs $\cap_i \{h_i \leq b_i\}$ in terms of F .

Lemma 19. *Suppose $\text{Td}(\bar{h}_i, \bar{h}_j) \leq \pi/2$ for all i and j , and let $F = (h_0, \dots, h_k)$. If $x \in \widetilde{M}$ and $b \in \mathbb{R}^{k+1}$ then*

$$(77) \quad |F(x) - F(p(b, x))|_\infty \leq d(x, p(b, x)) \leq |F(x) - b|_1 \leq (k+1)|F(x) - b|_\infty.$$

Proof. The left inequality is true because F is a contraction in the sup norm and the right inequality is obvious. Now we will prove the middle inequality. Let α be the piecewise geodesic path that starts at $x_0 := x$, goes along $-\nabla h_0$ for a time $\max\{h_0(x_0) - b_0, 0\}$ to a point x_1 , then goes along $-\nabla h_1$ for a time $\max\{h_1(x_1) - b_1, 0\}$ to a point x_2 , and so on. Since the angle between $-\nabla h_i$ and $-\nabla h_j$ is $\leq \pi/2$, all the coordinates h_i of F along this path are monotone decreasing. So, the path α starts at the point x and ends at a point x_{k+1} inside the intersection of horoballs $\cap_i \{h_i \leq b_i\}$. The monotonicity also implies that

$$\text{length}(\alpha) = \sum \max\{h_i(x_i) - b_i, 0\} \leq \sum |h_i(x_0) - b_i| = |F(x) - b|_1.$$

□

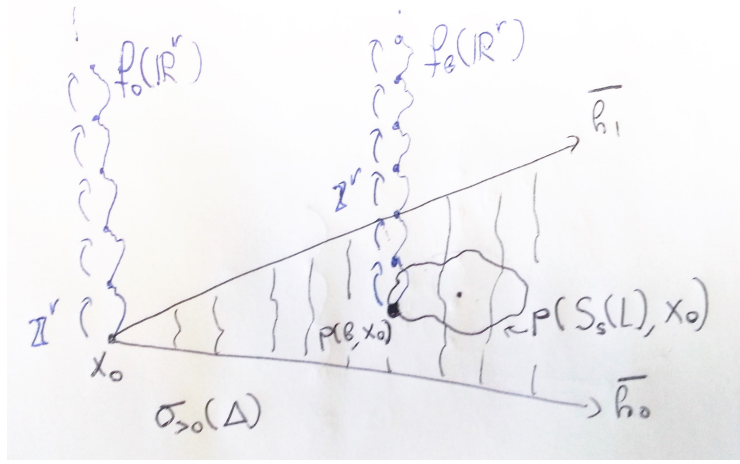


FIGURE 9.

Main ingredients. We show five things relating F, p, f_b, σ , and W .

- (1) For a Busemann simplex σ , the restriction

$$F|_{\sigma_{>0}(\Delta)}: \sigma_{>0}(\Delta) \rightarrow W$$

is a homeomorphism with inverse $p(\cdot, x_0)$.

- (2) All boundary points of W , except for $\{0\}$, are images of degenerate points, i.e.

$$F(\sigma_{>0}(\partial\Delta)) \supset \partial W \setminus \{0\}.$$

- (3) For any point $b \in W$,

$$F(f_b(0)) = b.$$

- (4) There is a constant $L > 0$ so that for every $b \in W$,

$$F(f_b(\mathbb{R}^r)) \text{ is contained in } B_b(L).$$

- (5) If σ is a non-degenerate, then W contains arbitrarily large balls.

Proof of Theorem 18 given (3)-(5). Property (5) implies we can find a punctured ball $B_s^*(2L) := B_s(2L) \setminus \{s\}$ in W . Property (4), \mathbb{Z}^r -equivariance of f_b , and \mathbb{Z}^r -invariance of F implies we can form the composition

$$(78) \quad \begin{array}{ccccc} S_s(L) \times \mathbb{T}^r & \rightarrow & F^{-1}(B_s^*(2L))/\mathbb{Z}^r & \rightarrow & B_s^*(2L) \times \widetilde{M}/\mathbb{Z}^r, \\ (b, \bar{z}) & \mapsto & \overline{f_b(z)} & \mapsto & (F(f_b(z)), \overline{f_b(z)}), \end{array}$$

which we will call (g_1, g_2) . Note that $g_1|_{b \times \mathbb{T}^r}$ is null-homotopic, since it lands in $B_b(L)$, while $g_2|_{b \times \mathbb{T}^r} = \overline{f_b}$ is a homotopy equivalence, since f_b is \mathbb{Z}^r -equivariant. In addition, property (3) means that $g_1|_{S_s(L) \times 0}$ is the identity map, while $g_2|_{S_s(L) \times 0}$ is null-homotopic, since it lifts to \widetilde{M} . Together, these imply that (g_1, g_2) is an isomorphism on homology (and thus also cohomology) in dimensions $< k + r$. Thus, it is also an isomorphism in

dimension $k + r$.²³ So, $F^{-1}(B_s^*(2L))/\mathbb{Z}^r$ has a non-trivial $(r + k)$ -homology class. It is an open submanifold of $\widetilde{M}/\mathbb{Z}^r$, so \widetilde{M} has dimension $\geq r + k + 1$.

8.3. Proof of properties (1) and (3). First we show

$$(79) \quad \sigma_R(t) = p(F(\sigma_R(t)), x_0).$$

Proof. Let $b := F(\sigma_R(t))$. The intersection of horoballs $\cap_i \{h_i \leq b_i\}$ contains $\sigma_R(t)$ and is contained in the sublevel set $\{h_0 t_0 + \dots + h_k t_k \leq b_0 t_0 + \dots + b_k t_k\}$. Since $\sigma_R(t)$ is the unique closest point to x_0 in this sublevel set, it is also the unique closest point to x_0 on the intersection of horoballs. \square

This shows that $F|_{\sigma_{>0}(\Delta)}$ is a homeomorphism onto its image, and $p(\cdot, x_0)$ is its inverse. This proves (1). Property (3) is a direct consequence.

Remark. For any $y \in \sigma_{>0}(\Delta)$ we have shown that $y = p(F(y), x_0)$, so we can apply Lemma 19 in the special case $x = x_0$ and $b = F(y)$ to get

$$(80) \quad |F(y)|_\infty \leq d(x_0, y) \leq (k + 1)|F(y)|_\infty.$$

8.4. Proof of property (2). By Corollary 17 and property (1), on the open subset $U \subset \mathbb{R}^+ \times \Delta^k$ the map $U \rightarrow \mathbb{R}^{k+1}$, $(R, t) \mapsto F(\sigma_R(t))$ is injective. So, by invariance of domain, it is a homeomorphism onto its image. Therefore, the image of the non-degenerate points $F(\sigma_{>0}(\Delta) \setminus \sigma_{>0}(\partial\Delta))$ is open, so it is contained in the interior of W . So, $W \cap \partial W$ consists entirely of the images of degenerate points $F(\sigma_{>0}(\Delta))$. It is not hard to see that $W \cap \partial W = \partial W \setminus \{0\}$.

8.5. Proof of property (4). Let L be any constant greater than the diameter of the image of the fundamental domain $f([0, 1]^r)$. Since f is \mathbb{Z}^r -equivariant, $f(\mathbb{R}^r)$ is contained in the L -neighborhood of $\mathbb{Z}^r x_0$. So, since $F(p(b, \cdot))$ is a \mathbb{Z}^r -invariant contraction, $F(f_b(\mathbb{R}^r)) = F(p(b, f(\mathbb{R}^r)))$ is contained in the L -neighborhood of $F(p(b, \mathbb{Z}^r x_0)) = b$.

8.6. Proof of property (5): If σ is non-degenerate then W is fat.

There is a sequence $R_i \rightarrow \infty$ so that the maps $\{\sigma_{R_i}\}$ converge to σ in the \angle_{x_0} -metric on the sphere at infinity ∂_∞ . Consequently, if σ is non-degenerate then we can find a constant $c > 0$ and arbitrarily large R so that

$$(81) \quad \text{there is a point } x \in \sigma_R(\Delta) \text{ with } \angle_{x_0}(x, \sigma_R(\partial\Delta)) > c.$$

Since \widetilde{M} is non-positively curved, triangle comparison implies that for the Riemannian metric $d := d_{\widetilde{M}}$ we have

$$(82) \quad d(x, \sigma_R(\partial\Delta)) > cR.$$

The dependence of σ_R on the parameter R is controlled by Corollary 8. It implies, for $\delta < 1$, that $\sigma_{[(1-\delta)R, (1+\delta)R]}(\partial\Delta)$ is contained in the $\sqrt{3\delta}R$ -neighborhood of $\sigma_R(\partial\Delta)$. Using this, we strengthen (82) (see Figure 10) to

²³One sees this, for instance, using cup products in cohomology.

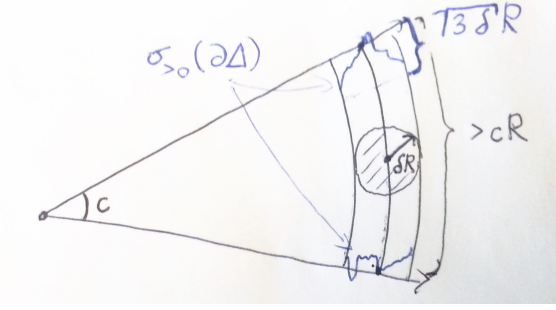


FIGURE 10.

the statement that there is a $\delta > 0$ ²⁴ and arbitrarily large R so that

$$(83) \quad \text{there is a point } x \in \sigma_R(\Delta) \text{ with } d(x, \sigma_{>0}(\partial\Delta)) > \delta R.$$

Next, we need to control how much F can shrink things.

Lemma 20. *Pick $x \in \sigma_R(\Delta)$ and $y \in \sigma_{>0}(\Delta)$ with $d(F(x), F(y)) \leq \frac{R}{k+2}$. Then, there is a constant C depending only on k so that*

$$(84) \quad d(F(x), F(y)) \geq C \cdot \frac{d(x, y)^2}{R}.$$

Proof. Let $b \in \mathbb{R}^{k+1}$ be any point satisfying the inequalities $b_i \leq h_i(x)$ for all i . Set $z := p(b, x_0)$. The main point is to prove inequality (89), below.

Note that z is in the intersection of horoballs $\cap_i \{h_i \leq h_i(x)\}$ and x is the closest point to x_0 in this intersection. Since $\cap_i \{h_i \leq h_i(x)\}$ is convex, this implies $\angle_x(x_0, z) \geq \pi/2$. Comparing with the Euclidean triangle with geodesic sides $[x_0, x]$ and $[x, z]$ meeting at an angle $\angle_x(x_0, z)$, we see that

$$(85) \quad d(x, z)^2 \leq d(x_0, z)^2 - d(x_0, x)^2.$$

Since $p(b, x)$ is a point on the intersection of horoballs $\cap_i \{h_i \leq b_i\}$ and z is the closest point to x_0 in this intersection, $d(x_0, z) \leq d(x_0, p(b, x))$ and so

$$(86) \quad d(x, z)^2 \leq d(x_0, p(b, x))^2 - d(x_0, x)^2.$$

Using the triangle inequality and Lemma 19, we get

$$(87) \quad d(x, z)^2 \leq (d(x_0, x) + d(x, p(b, x)))^2 - d(x_0, x)^2,$$

$$(88) \quad \leq (d(x_0, x) + (k+1)d(F(x), b))^2 - d(x_0, x)^2,$$

which can be rewritten as

$$(89) \quad d(x, z)^2 \leq d(x_0, x)d(F(x), b) \left(2(k+1) + (k+1)^2 \frac{d(F(x), b)}{d(x_0, x)} \right).$$

Now, we specify b to be

$$(90) \quad b_i := \min\{h_i(x), h_i(y)\}.$$

²⁴We can write down an explicit δ that is quadratic in c .

Since the metric d is defined via the supremum norm, we have

$$(91) \quad \max\{d(F(x), b), d(b, F(y))\} = d(F(x), F(y)) \leq \frac{R}{(k+2)}.$$

Recall that $d(x_0, x) = R$. So, inequality (89) implies there is a constant A , depending only on k , for which $d(x, z)$ is bounded by $\sqrt{AR \cdot d(F(x), F(y))}$. We will use (89) (with x replaced by y) to get a similar constraint on $d(y, z)$. To do this, we need to bound $d(x_0, y)$ from above and below in terms of R . First, note that (80) (with y replaced by x) implies that

$$(92) \quad \frac{R}{k+1} \leq |F(x)|_\infty \leq R.$$

Since $||F(y)|_\infty - |F(x)|_\infty| \leq d(F(x), F(y)) \leq R/(k+2)$, this implies

$$(93) \quad \left(\frac{1}{k+1} - \frac{1}{k+2}\right) R \leq |F(y)|_\infty \leq \left(1 + \frac{1}{k+2}\right) R.$$

Now (80) gives the two-sided bound we want:

$$(94) \quad \frac{1}{(k+1)(k+2)} R \leq d(x_0, y) \leq \frac{(k+1)(k+3)}{k+2} R.$$

Now, we look at inequality (89) (with x replaced by y) and note that the bound (94) implies there is a constant $A' \geq A$, depending only on k , for which $d(y, z)$ is bounded by $\sqrt{A'R \cdot d(F(x), F(y))}$. So

$$\begin{aligned} d(x, y)^2 &\leq (d(x, z) + d(z, y))^2, \\ &\leq d(x, z)^2 + d(z, y)^2 + 2d(x, z)d(z, y), \\ &\leq 4A'R \cdot d(F(x), F(y)). \end{aligned}$$

This completes the proof of the lemma. \square

Finishing the proof of Property (5). Make δ smaller, if necessary, so that $C\delta^2 R \leq R/(k+2)$, and look at the ball $B := B_{F(x)}(C\delta^2 R)$. By (83), any degenerate point $y \in \sigma_{>0}(\partial\Delta) \cap F^{-1}(B)$ satisfies $d(x, y) > \delta R$. But then, by Lemma 20, $d(F(x), F(y)) > C\delta^2 R$, contradicting that $F(y) \in B$. So, there are no such points y . Therefore, the ball B does not contain any images of degenerate points $F(\sigma_{>0}(\partial\Delta))$ and so by property (2) it does not contain any of the boundary points $\partial W \setminus \{0\}$. It also does not contain $\{0\}$.²⁵ So the ball B contains $F(x) \in W$ but does not contain any of the boundary points ∂W . Thus B is contained entirely inside W . Since R can be taken arbitrarily large, this shows that W is fat.

²⁵Because $\{0\}$ is a limit of other boundary points. Or, more explicitly, because

$$d(0, F(x)) \geq \frac{d(x_0, x)}{k+1} > C\delta^2 R.$$

9. DIVERGENT GEODESIC RAYS AND SECTORS

9.1. The thin limit set and divergent rays. The *thin limit set*

$$\Delta := \{\xi \in \partial_\infty \mid \xi = r(\infty) \text{ for some } r : [0, \infty) \rightarrow \widetilde{M} \text{ with } p(r(t)) \rightarrow \infty\}$$

consists of endpoints $r(\infty)$ of geodesic rays r whose projections diverge.

Theorem 21. *Suppose $N := \langle \gamma_1, \dots, \gamma_r \rangle$ is a finitely generated group. Pick a point $\xi \in \text{Fix}(N)$. Then ξ is in the thin limit set if either*

- *there is a point $\eta \in \text{Fix}^0(C_N)$ with $\text{Td}(\eta, \xi) < \pi/2$, or*
- *$\xi \in \text{Fix}(C_N)$, N has a parabolic γ , and \widetilde{M} has negative curvature.*²⁶

Proof. Suppose ξ is not in the thin limit set. Then, the projection $p \circ r$ of a geodesic ray $r : [0, \infty) \rightarrow \widetilde{M}$ pointing to ξ does not diverge, so there is a sequence of times $t_i \rightarrow \infty$ and elements $g_i \in \Gamma$ so that $\{g_i r(t_i)\}$ converges to a point x_0 in \widetilde{M} . Let $\varepsilon_i := d(x_0, g_i r(t_i))$.

Claim: If $\text{Td}(\xi, \eta) < \pi/2$ then for large enough j , the element $g_j^{-1} g_i$ does not preserve the horosphere $H(r(t_i), \eta)$. As $j \rightarrow \infty$ we have

$$(95) \quad d(g_j^{-1} g_i r(t_i), r(t_j)) = d(g_i r(t_i), g_j r(t_j)) \rightarrow d(g_i r(t_i), x_0) = \varepsilon_i.$$

If $\text{Td}(\xi, \eta) < \pi/2$ then $\angle_{r(t)}(\eta, \xi) < \pi/2$, so the geodesic ray $[r(t_i), \xi)$ keeps going further into the interior of the horoball $HB(r(t_i), \eta)$. So, for large enough t_j we have $B_{2\varepsilon_i}(r(t_j)) \subset HB(r(t_i), \eta)$. So, equation (95) implies that for large enough j , the point $g_j^{-1} g_i r(t_i)$ is in the interior of the horoball $HB(r(t_i), \eta)$. So, for large j , we see that $g_j^{-1} g_i$ does not preserve $H(r(t_i), \eta)$.

Claim: After passing to a subsequence of $\{g_i\}$, we have $g_j^{-1} g_i \in C_N$.

For any element γ , the triangle inequality²⁷ implies

$$(96) \quad |d_{g_i \gamma g_i^{-1}}(x_0) - d_{g_i \gamma g_i^{-1}}(g_i r(t_i))| \leq 2\varepsilon_i.$$

If γ fixes $r(\infty) = \xi$ we also get

$$(97) \quad d_{g_i \gamma g_i^{-1}}(g_i r(t_i)) = d_\gamma(r(t_i)),$$

$$(98) \quad \leq d_\gamma(r(0)),$$

so that $\{d_{g_i \gamma g_i^{-1}}(x_0)\}_{i=1}^\infty$ is bounded. Thus, there are only finitely many different conjugates in the sequence $\{g_i \gamma g_i^{-1}\}_{i=1}^\infty$. After passing to a subsequence, we may assume that all the conjugates are the same, i.e. that

$$(99) \quad g_1 \gamma g_1^{-1} = g_2 \gamma g_2^{-1} = \dots,$$

and consequently that $g_j^{-1} g_i$ commutes with γ .

If $\gamma_1, \dots, \gamma_r$ is a finite collection of elements fixing ξ , then we can do the above process for each one of them and, after taking subsequences, get a

²⁶We won't use this second bullet point anywhere in the paper.

²⁷For any isometry ρ , it follows from triangle inequality that $|d_\rho(x) - d_\rho(y)| \leq 2d(x, y)$.

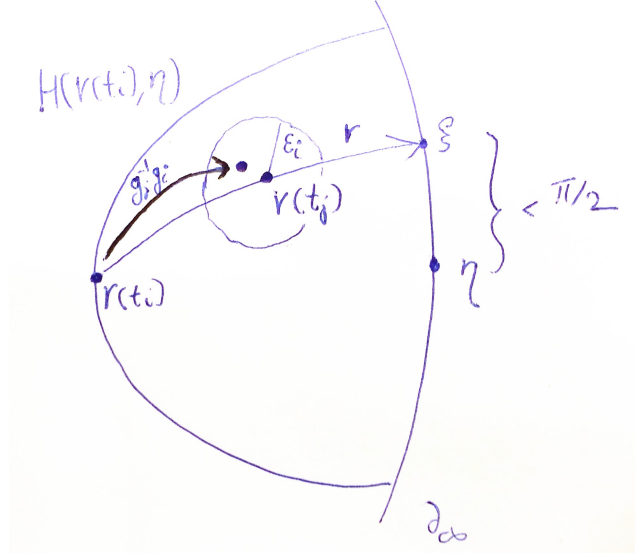


FIGURE 11.

sequence $\{g_i\}$ for which $g_j^{-1}g_i$ commutes with the group $N = \langle \gamma_1, \dots, \gamma_r \rangle$. This proves the second claim.

The two claims together show that if $\xi \notin \Delta$ then there is no point $\eta \in \text{Fix}^0(C_N)$ with $\text{Td}(\xi, \eta) < \pi/2$. This proves the first bullet of the theorem.

No flat regions in negative curvature. Let $s : [0, \infty) \rightarrow \widetilde{M}$ be the geodesic ray $[g_1^{-1}x_0, \xi)$. To prove the second bullet, we will show that for elements $\gamma \in N$, displacement of γ is constant along s . So, the geodesic rays s and γs are parallel. If γ is parabolic then the rays s and γs are also disjoint. So, joining them via geodesics forms a flat (half) strip. There are no such strips in negative curvature, so we get the second bullet once we prove the following.

Claim: The displacement of every $\gamma \in N$ is constant along s . First note that by our hypothesis, the elements $g_1^{-1}g_i \in C_N$ fix ξ , so the geodesic rays $g_1^{-1}g_i r$ and s travel to the same point ξ at infinity. Thus

$$(100) \quad d(g_1^{-1}g_i r(t + t_i), s(t)) \leq d(g_1^{-1}g_i r(t_i), g_1^{-1}x_0),$$

$$(101) \quad = d(g_1^{-1}g_i r(t_i), s(0)),$$

$$(102) \quad = \varepsilon_i.$$

Now, look at an element $\gamma \in N$. Since $g_1^{-1}g_i$ commutes with γ , we have

$$(103) \quad d_\gamma(g_1^{-1}g_i r(t + t_i)) = d_\gamma(r(t + t_i)).$$

This, together with (100), implies that

$$(104) \quad |d_\gamma(r(t+t_i)) - d_\gamma(s(t))| = |d_\gamma(g_1^{-1}g_i r(t+t_i)) - d_\gamma(s(t))|$$

$$(105) \quad \leq 2\varepsilon_i.$$

Since $\varepsilon_i \rightarrow 0$ and $t_i \rightarrow \infty$ as $i \rightarrow \infty$, taking the limit of the above inequality as $i \rightarrow \infty$ gives

$$(106) \quad d_\gamma(s(t)) = \lim_{i \rightarrow \infty} d_\gamma(r(t+t_i)) = \lim_{t \rightarrow \infty} d_\gamma(r(t)).$$

This proves that the displacement of γ is constant along s , and thus proves the second bullet. \square

9.2. Divergent sectors. We will also need a version of Theorem 21 for compact families of geodesic rays. Pick a basepoint $y_0 \in \widetilde{M}$ and let $K \subset \partial_\infty$ be a compact subset. For any $t \geq 0$ and $\xi \in K$, let $c_t(\xi)$ be the point obtained by going for a time t along the geodesic ray $r_\xi := [y_0, \xi)$. For each time t , this defines a map

$$(107) \quad c_t : K \rightarrow \widetilde{M}.$$

Remark. The maps $\{c_t\}_{t \geq 0}$ parametrize points on the geodesic cone $y_0 * K$.

The proof of Theorem 21 generalizes to give divergent sectors.

Proposition 22. *Suppose $N := \langle \gamma_1, \dots, \gamma_r \rangle$ is a finitely generated group. Let $K \subset \text{Fix}(N)$ be a compact set, and suppose*

- *there is $\eta \in \text{Fix}^0(C_N)$ so that for all $\xi \in K$ we have $\text{Td}(\eta, \xi) < \pi/2$.*

Pick a basepoint $y_0 \in \widetilde{M}$ and let $c_t : K \rightarrow \widetilde{M}$ be the maps defined above. Then, for every $\epsilon > 0$ there is T so that

- *$c_t(K) \subset \widetilde{M}_{<\epsilon}$ for all $t \geq T$.*

The proof is a similar to the proof of the first bullet of Theorem 21.

Proof. Suppose the conclusion is not true. Then there are times $t_i \rightarrow \infty$, elements $g_i \in \Gamma$ and points at infinity $\xi_i \in K$ so that $\{g_i r_{\xi_i}(t_i)\}$ converges to a point x_0 in \widetilde{M} . We will simplify notation slightly and write r_i for the geodesic ray r_{ξ_i} . Let D be the diameter of the set $\{g_i r_i(t_i)\}_{i=1}^\infty$. Since K is compact, there is a constant $\delta > 0$ so that $\text{Td}(\xi, \eta) \leq \pi/2 - \delta$ for every $\xi \in K$. Let h be the horofunction of η , normalized so that $h(y_0) = 0$. Then,

$$(108) \quad h(r_\xi(t)) \leq -t \cdot \sin \delta.$$

Moreover,

$$(109) \quad d(g_j^{-1}g_i r_i(t_i), r_j(t_j)) = d(g_i r_i(t_i), g_j r_j(t_j)) \leq D,$$

implies that

$$(110) \quad h(g_j^{-1}g_i r_i(t_i)) \leq h(r_j(t_j)) + D \leq -t_j \cdot \sin \delta + D.$$

Since $t_j \rightarrow \infty$, for sufficiently large j this is strictly less than $h(r_i(t_i))$. So, for large j the horofunction h is not $g_j^{-1}g_i$ invariant and consequently

- for large j , the horosphere $H(r_i(t_i), \eta)$ is not preserved by $g_j^{-1}g_i$.

On the other hand, arguing as in the second claim of the proof of Theorem 21, (and noting that $d(x_0, g_i r_i(t_i)) \leq D$) we see that for any element γ that

$$(111) \quad |d_{g_i \gamma g_i^{-1}}(x_0) - d_{g_i \gamma g_i^{-1}}(g_i r_i(t_i))| \leq 2D.$$

If γ fixes all the points $\{r_i(\infty)\}_i \in K$, we also get that

$$(112) \quad d_{g_i \gamma g_i^{-1}}(g_i r_i(t_i)) = d_\gamma(r_i(t_i)) \leq d_\gamma(r_i(0)) = d_\gamma(y_0).$$

So, for such γ , we see that the set $\{d_{g_i \gamma g_i^{-1}}(x_0)\}_{i=1}^\infty$ is bounded. Now, arguing exactly as in Theorem 21 we conclude that,

- after passing to a subsequence of $\{g_i\}$, we have $g_j^{-1}g_i \in C_N$.

Putting these two bullets together, we conclude that $\eta \notin \text{Fix}^0(C_N)$, which contradicts the hypotheses of the proposition. \square

9.3. Free abelian groups. All we will use in the rest of the paper is the following corollary.

Corollary 23. *If A is free abelian, then $\text{Fix}^0(C_A)$ is in the thin limit set. More generally, for any $\epsilon > 0$, basepoint $y_0 \in \widetilde{M}$ and compact $K \subset \text{Fix}^0(C_A)$, we have $c_t(K) \subset \widetilde{M}_{<\epsilon}$ for large enough t .*

Proof. Let $\xi \in \text{Fix}^0(C_A)$. Since A is abelian, ξ is automatically in $\text{Fix}(A)$. So, applying the first bullet of Theorem 21 with $\eta = \xi$, we see that ξ is in the thin limit set.

Now we prove the second part of the corollary. Since K is compact, we can cover it by finitely many balls $\{B_{\pi/2-\delta}(\eta_i)\}_{i=1}^r$ centered at points $\eta_i \in \text{Fix}^0(C_A)$. Apply Proposition 22 to the compact sets $K_i := K \cap \overline{B}_{\pi/2-\delta}(\eta_i)$ and points η_i to get constants $\{T_i\}_{i=1}^r$ so that $c_t(K_i)$ is in $\widetilde{M}_{<\epsilon}$ for $t \geq T_i$. Then $c_t(K) = c_t(K_1 \cup \dots \cup K_r) \subset \widetilde{M}_{<\epsilon}$ for $t \geq \max_i T_i$. \square

Remark. One might try to get the second part of Corollary 23 directly from the first part and compactness of K . The problem with this approach is that one does not know whether the function

$$(113) \quad \tau_\epsilon : K \rightarrow [0, \infty),$$

$$(114) \quad \tau_\epsilon(\xi) := \inf \left\{ T \mid c_t(\xi) \subset \widetilde{M}_{<\epsilon} \text{ for all } t \geq T \right\}$$

is continuous. It is apriori possible that $\xi_i \rightarrow \xi$ and $\tau_\epsilon(\xi_i) \rightarrow \infty$, even though $\tau_\epsilon(\xi)$ is finite, which would prevent us from getting a uniform bound on τ_ϵ . Proposition 22 shows, for points $\{\xi_i\}$ in $\text{Fix}(N)$ that are sufficiently close to $\text{Fix}^0(C_N)$, that this troubling scenario does not occur.

9.4. A remark about closed manifolds. While our interest in this paper is in noncompact manifolds, it is instructive to note what the results of this section say when M is a *closed* manifold. For abelian groups $A < \pi_1 M$, we conclude that the centralizer C_A does not preserve any horosphere. More generally, for any finitely generated subgroup $N < \pi_1 M$, Theorem 21 implies that $\text{Td}(\text{Fix}^0(C_N), \text{Fix}(N)) \geq \pi/2$.

10. THE HOROSPHERICAL DEFORMATION

In this section we prove Theorems 1 and 6. The constants $\varepsilon < \epsilon$ are the ones introduced in Section 3. Let P be a *finite* polyhedron and $\varphi : P \rightarrow \widetilde{M}_{<\epsilon}$ a map into the ϵ -thin part. Since this thin part is a product $\widetilde{M}_{<\epsilon} = \partial\widetilde{M}_{<\epsilon} \times [0, \infty)$, we may begin by pushing φ further into the ε -thin part $\widetilde{M}_{<\varepsilon}$. So, from now on we will assume that φ lands in the ε -thin part, i.e. $\varphi : P \rightarrow \widetilde{M}_{<\varepsilon}$. Also, after refining the triangulation of P , the image of each simplex $\varphi(\sigma)$ in P has diameter $< \varepsilon$. Now, we describe a deformation $\varphi_t(P)$ pushing $\varphi(P)$ to infinity in the ϵ -thin part by pushing all of the simplices in a compatible way.

Remark. This deformation will be described on each simplex in terms of the *second barycentric subdivision* P_2 of the polyhedron. The reason is that we can connect nilpotent groups Γ_0 and Γ_1 corresponding to vertices of a simplex in P by inclusions of nilpotent groups via

$$(115) \quad \Gamma_0 < \Gamma_{01} > \Gamma_1,$$

which is a path in the barycentric subdivision P_1 , but we can connect the centers Z_0 and Z_1 by inclusions of *abelian* groups

$$(116) \quad Z_0 < \langle Z_0, Z_{01} \rangle > Z_{01} < \langle Z_{01}, Z_1 \rangle > Z_1,$$

which is a path in the second barycentric subdivision P_2 .

10.1. Nilpotent groups and abelian groups. For each vertex v of P we have a nilpotent group $\Gamma_{h(v)}$, defined by (6). The vertices of the first barycentric subdivision P_1 are simplices $\sigma = v_0 * \dots * v_k$ of P , and for each of these simplices we also have a nilpotent group $\Gamma_{\varphi(\sigma)} := \langle \Gamma_{\varphi(v_0)}, \dots, \Gamma_{\varphi(v_k)} \rangle$. The vertices of the second barycentric subdivision P_2 are chains $\sigma_0 \subset \dots \subset \sigma_k$ of simplices in P_1 , giving a chain of nilpotent groups $\Gamma_{\sigma_0} < \dots < \Gamma_{\sigma_k}$. Corresponding to this chain is the abelian²⁸ group generated by the centers

$$(117) \quad A_{\sigma_0 \subset \dots \subset \sigma_k} := \langle Z_{\sigma_0}, \dots, Z_{\sigma_k} \rangle.$$

So, for each vertex $v \in P_2$ we have an abelian group A_v . For each ordered k -simplex $v_0 * \dots * v_k$ in P_2 we have a chain of abelian groups $A_{v_0} < \dots < A_{v_k}$.

²⁸It is abelian because Z_{σ_i} centralizes the group $\langle Z_{\sigma_0}, \dots, Z_{\sigma_{i-1}} \rangle \subset \Gamma_{\sigma_i}$.

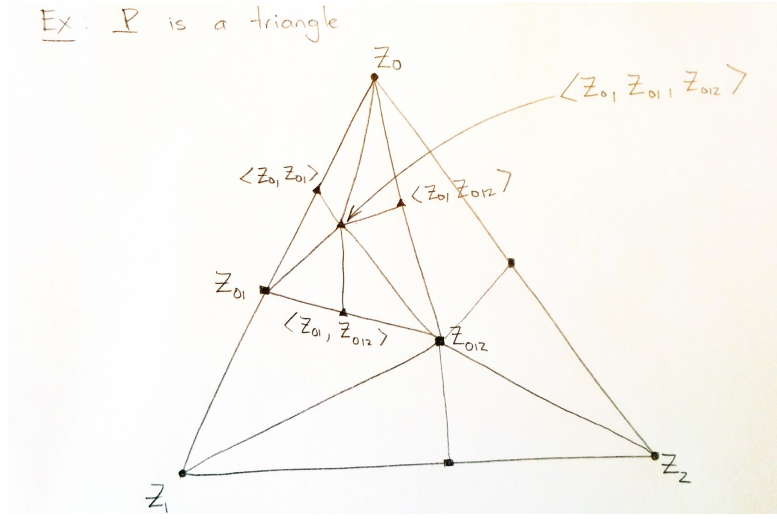


FIGURE 12.

10.2. **Virtual equivalence classes.** Now, let

$$(118) \quad [A_v] := \bigcap_{A_w >_{\text{vir}} A_v} A_w$$

be the intersection of all groups A_w that virtually contain A_v .

- Since P is a finite complex, this is a finite index subgroup of A_v .
- If A_v and A_w are virtually equivalent then $[A_v] = [A_w]$.
- Any chain $A_{v_0} < \dots < A_{v_k}$ gives²⁹ a chain $[A_{v_0}] < \dots < [A_{v_k}]$.

10.3. Mapping P into the thin limit set via Busemann simplices.

Since A_v contains some center Z_σ , it contains a parabolic element ζ (see end of Section 3). Any positive power ζ^k is parabolic (Lemma 6.6 of [4]), so the finite index subgroup $[A_v]$ also contains parabolic isometries. For each distinct abelian group $[A_v]$ we can pick a combined displacement function $f_v := d_{[A_v]^0} + d_{[A_v]^+}$ as described in subsection 5.10. This function is $C_{[A_v]}$ -invariant and convex. Moreover, since $[A_v]$ contains a parabolic, f_v it does not attain its infimum. Thus, there is a corresponding set of points at infinity ∂f_v . Pick one such point $\xi_v \in \partial f_v$ for each distinct abelian group $[A_v]$. For any group A_w that is virtually equivalent to A_v , we let $\xi_w = \xi_v$ be the same point. For any chain $[A_{v_0}] < \dots < [A_{v_k}]$ the set of points $\{\xi_{v_0}, \dots, \xi_{v_k}\}$ has diameter $\leq \pi/2$.³⁰ So we can form a Busemann simplex with vertices $\xi_{v_0}, \dots, \xi_{v_k}$,

$$(119) \quad \blacktriangle : v_0 * \dots * v_k \rightarrow \text{Fix}^0(C_{[A_{v_k}]}).$$

Its image is in the thin limit set by Corollary 23.

²⁹This is the reason why we use virtual containment instead of virtual equality in (118).

³⁰Because the larger set $\partial f_{v_0} \cup \dots \cup \partial f_{v_k}$ has diameter $\leq \pi/2$ by Corollary 11.

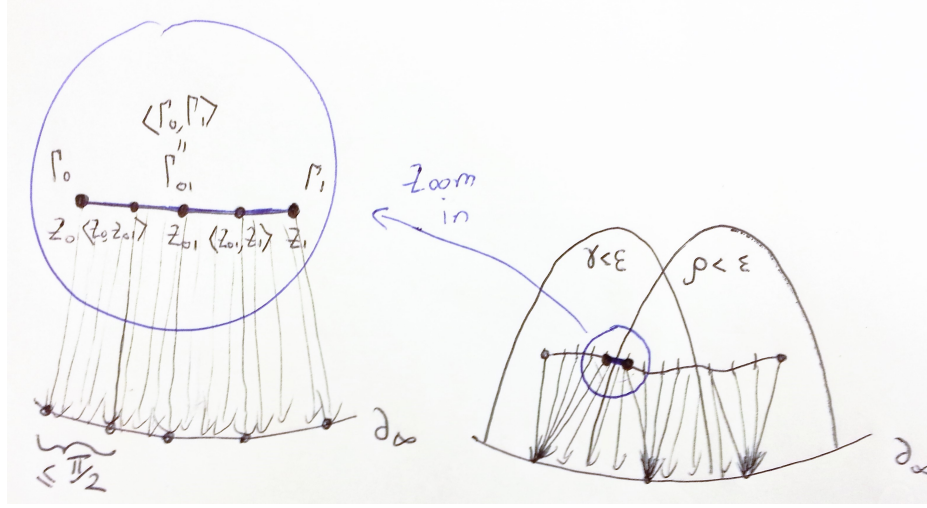


FIGURE 13.

Lemma 24. *If \blacktriangle is non-degenerate, then $k < \lfloor n/2 \rfloor$.*

Proof. If \blacktriangle is a non-degenerate Busemann k -simplex, then

- by Theorem 18 we have $k \leq n - 1 - \text{rank}[A_{v_k}]$, and
- all the groups in the chain $[A_{v_0}] < \dots < [A_{v_k}]$ are distinct, so the largest group $[A_{v_k}]$ has rank $\geq k + 1$.

Together, these imply $k \leq n - k - 2$ and consequently $k \leq \lfloor n/2 \rfloor - 1$. \square

Remark. Let r be the maximum rank of abelian subgroups in Γ . Then the Busemann k -simplex \blacktriangle has at most r distinct vertices, so if it is non-degenerate then we must have $k < r$. If the abelian subgroups in Γ have small rank then this is sometimes a stronger bound than the one stated in Lemma 24. One of the main points of this paper is that even if there are abelian subgroups of high rank, we still get the half-dimensional bound because of Theorem 18.

In Section 6 we showed that we can construct Busemann simplices in a compatible way so as to get a continuous map to the thin limit set

$$(120) \quad \rho : P \rightarrow \Delta$$

whose restriction to any simplex $v_0 * \dots * v_k$ of P_2 is a Busemann simplex of the form \blacktriangle . Moreover, the map ρ is not just continuous but *Lipschitz* in the \angle_{x_0} -metric on the sphere at infinity ∂_∞ .

10.4. The image of ρ is half-dimensional. Every simplex is a union of its non-degenerate sub-simplices, so Lemma 24 implies that the image of ρ is the image of the $(\lfloor n/2 \rfloor - 1)$ -skeleton:

$$(121) \quad \rho(P) = \rho(P^{(\lfloor n/2 \rfloor - 1)}).$$

This, together with the Lipschitzness of ρ can be used to show that the image $\rho(P)$ is “half-dimensional” in several different senses. For instance, Lipschitz maps do not increase Hausdorff dimension, so (121) implies

$$(122) \quad \text{Hausdorff dim}_{(\partial_\infty, \angle_{x_0})}(\rho(P)) \leq \lfloor n/2 \rfloor - 1.$$

Remark. Since as a metric space $(\partial_\infty, \angle_{x_0})$ is just a round sphere, this also bounds the topological dimension of the image (see section 5 of [9])

$$(123) \quad \text{topdim}(\rho(P)) < \lfloor n/2 \rfloor.$$

10.5. A copy of $\rho(P)$ in the R -sphere centered at x_0 . The round sphere $(\partial_\infty, \angle_{x_0})$ is bi-Lipschitz homeomorphic to the R -sphere $(S_{x_0}(R), d_{\widetilde{M}})$, via the exponential map³¹. Under this homeomorphism, $\rho(x)$ is taken to the point $c_R(x)$ obtained by flowing for a time R along the geodesic ray $[x_0, \rho(x))$. For $R > 0$, this gives a Lipschitz map

$$(124) \quad c_R : P \rightarrow \widetilde{M}$$

whose image $c_R(P)$ is homeomorphic to $\rho(P)$. As before, the image of c_R is the image of the $(\lfloor n/2 \rfloor - 1)$ -skeleton so the previous section implies $c_R(P)$ has Hausdorff (and topological) dimension $< \lfloor n/2 \rfloor$. But, for us it is more useful to know that c_R is close to map that factors through a low-dimensional polyhedron.

Lemma 25. *For any $\delta > 0$, the map c_R is within δ of a map that factors through a polyhedron of dimension $< \lfloor n/2 \rfloor$.*

The argument is fairly standard,³² so we will be brief. Let $k := \lfloor n/2 \rfloor - 1$.

Proof sketch. Cover the image of c_R by finitely many δ -balls, and let N be the nerve of this cover. Use smooth partitions of unity to define a Lipschitz map $c_R(P) \xrightarrow{p} N$, and use geodesic simplices to build a map $N \xrightarrow{\tau} \widetilde{M}$. The composition $\tau \circ p$ of these two maps is not quite the identity, but it is within δ of the identity. Now, the map $p \circ c_R : P \rightarrow N$ is Lipschitz, and its image is contained in the image of the k -skeleton $p \circ \sigma_R(P^{(k)})$. So, if σ is a simplex in N of dimension $> k$ then there is a point x in the interior of σ that is not in the image $p \circ c_R(P)$.³³ So, we can take the radial projection to the boundary $\partial\sigma$ to get a Lipschitz³⁴ map that misses the interior of σ . Iterating this process, we finally get a map to the k -skeleton of the nerve $p' : c_R(P) \rightarrow N^{(k)}$ so that the composition $\tau \circ p'$ is still within δ of the identity map. Thus, $\tau \circ p' \circ c_R$ is a map that factors through a k -dimensional polyhedron $N^{(k)}$ and is within δ of the map c_R . \square

³¹The exponential map is a diffeomorphism and diffeomorphisms of closed Riemannian manifolds are, of course, bi-Lipschitz homeomorphisms.

³²See, e.g. section 5 of [9] for this kind of argument in a CAT(1) setting.

³³This is because Lipschitz maps do not raise Hausdorff dimension.

³⁴Since the image misses the point x , it actually misses a small ball centered at x . The Lipschitz constant of the radial projection is bounded on the complement of this ball.

10.6. The finite time deformation φ_t . Let $\varphi_t(x)$ be the point obtained by flowing for a time t along the geodesic ray $[\varphi(x), \rho(x))$. This gives a map

$$(125) \quad \varphi_t : P \rightarrow \widetilde{M}.$$

Claim: The deformation φ_t stays in the ϵ -thin part of \widetilde{M} .

Proof. Let σ be a k -simplex in P_2 corresponding to a chain $A_0 < \dots < A_k$.

- There is an element $\zeta \in A_k$ such that $d_\zeta(\varphi(\sigma)) < \epsilon = (I_n!)3^{n+1}\epsilon$.

Proof. The group A_k contains the center of Γ_x for some point $x \in \varphi(\sigma)$. This center has a nontrivial element γ that can be expressed as a product of at most $(I_n!)3^n$ elements of S_x (see Lemma 8 of [1]). Since the vertices of $\varphi(\sigma)$ are in the ϵ -thin part and each simplex has diameter $< \epsilon$, the elements of S_x move $\varphi(\sigma)$ by at most 3ϵ . By the triangle inequality, ζ moves $\varphi(\sigma)$ by at most ϵ . \square

The group A_k fixes all of $\rho(\sigma) \subset \text{Fix}(C_{\lfloor A_k \rfloor})$. So, the displacement of the element ζ is $< \epsilon$ everywhere in the join $\varphi(\sigma)*\rho(\sigma)$. This proves the claim. \square

10.7. Deforming φ_t to the cone c_t inside $\widetilde{M}_{<\epsilon}$. We can deform φ_t to c_t by pushing $\varphi_t(x)$ to $c_t(x)$ along the geodesic segment connecting them.

Claim: For large enough t , this deformation is in the ϵ -thin part.

Proof. Since the rays $[\varphi(x), \rho(x))$ and $[x_0, \rho(x))$ have the same endpoint, the geodesic connecting $\varphi_t(x)$ to $c_t(x)$ has length $\leq D := \max_{x \in P} d(x_0, \varphi(x))$. It is easy to see there is $\epsilon' < \epsilon$ so that $d(M_{\geq \epsilon}, M_{< \epsilon'}) \geq D$ and consequently

$$(126) \quad d(\widetilde{M}_{\geq \epsilon}, \widetilde{M}_{< \epsilon'}) \geq D.$$

Since P_2 is a finite polyhedron and the image of each of its simplices $\rho(\sigma)$ is in some $\text{Fix}^0(C_{\lfloor A_v \rfloor})$, Corollary 23 implies the image of $c_t : P \rightarrow \widetilde{M}$ is in the ϵ' -thin part for large enough t . Then the D -neighborhood of $c_t(P)$ is in the ϵ -thin part by (126), and hence all the geodesic segments $\{[\varphi_t(x), c_t(x)]\}_{x \in P}$ lie completely inside the ϵ -thin part. This proves the claim. \square

In summary, we have deformed φ inside the ϵ -thin part to a Lipschitz map $c_t : P \rightarrow \widetilde{M}_{<\epsilon}$ whose image is contained in the image of the $(\lfloor n/2 \rfloor - 1)$ -skeleton (by subsection 10.5). This proves Theorem 6. Applying Lemma 25 to c_t gives a map $\widehat{\varphi} : P \rightarrow Q \rightarrow \widetilde{M}_{<\epsilon}$ that is δ -close to c_t and factors through a polyhedron Q of dimension $< \lfloor n/2 \rfloor$. For small enough δ , the geodesic homotopy between $\widehat{\varphi}$ and c_t stays within the ϵ -thin part. So, φ can be homotoped to $\widehat{\varphi}$ within the ϵ -thin part. This proves Theorem 1.

11. COROLLARIES

11.1. Asphericity in dimension less than 6. If M is ≤ 5 dimensional, then Q is a graph, so in this case $M_{<\epsilon}$ is aspherical.

11.2. Homology of the thin part. For general $n = \dim M$ it follows from Theorem 1 that

$$(127) \quad H_{\geq \lfloor n/2 \rfloor}(\widetilde{M}_{<\epsilon}) = 0.$$

Remark. Let r be the maximum rank of abelian subgroups of Γ . The proof of Theorem 1 actually shows that

$$(128) \quad H_{\geq \min\{r, \lfloor n/2 \rfloor\}}(\widetilde{M}_{<\epsilon}) = 0,$$

which is stronger than (127) when r is small. One of the interesting aspects of (127) is that we still have the half-dimensional bound even when there are large abelian subgroups³⁵.

11.3. Geometric dimension of the fundamental group. The fibre bundle $\widetilde{M}_{<\epsilon} \rightarrow (\widetilde{M}_{<\epsilon} \times M)/\Gamma \rightarrow M$ has total space homotopy equivalent to the closed $(n-1)$ -manifold $\partial := \partial M_{<\epsilon}$, and the base space is a $B\Gamma$, so the spectral sequence of this fibre bundle says

$$(129) \quad \bigoplus H_i(B\Gamma; H_{n-1-i}(\widetilde{M}_{<\epsilon})) \implies H_{n-1}(\partial) = \mathbb{Z}.$$

Together with (127), this implies $H_i(B\Gamma; H_{n-1-i}(\widetilde{M}_{<\epsilon}))$ is non-zero for some $n-1-i < \lfloor n/2 \rfloor$, that is $i \geq n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$. Consequently,

$$(130) \quad \text{gdim}(\Gamma) \geq \lceil n/2 \rceil.$$

Remark. If r is the maximum rank of an abelian subgroup of Γ then the same argument, using (128) instead of (127), implies that $\text{gdim}(\Gamma) \geq n - r$. On the other hand, we (of course) have $\text{gdim}(\Gamma) \geq r$ so it follows that

$$(131) \quad \text{gdim}(\Gamma) \geq \max\{r, n - r\}.$$

In the extreme case $\text{gdim}(\Gamma) = \lceil n/2 \rceil$ this implies Γ contains an abelian subgroup of rank $\lfloor n/2 \rfloor$.

11.4. Relation to action dimension of Γ . The following is sometimes called the action dimension conjecture for nonpositively curved manifolds.

Conjecture 26. *Suppose M is a complete, finite volume manifold of bounded nonpositive curvature with fundamental group $\Gamma = \pi_1 M$. Then any aspherical manifold with fundamental group Γ has dimension $\geq \dim M$.*

The conjecture is known for locally symmetric spaces and in some other cases (e.g. manifolds of pinched negative curvature), but not in general.

Now, it is well known that any d -dimensional $B\Gamma$ is homotopy equivalent to a $2d$ -dimensional manifold $B\Gamma$.³⁶ So, if Conjecture 26 is true, then one would have $2d \geq \dim M$ and therefore any $B\Gamma$ must have dimension $\geq \lceil n/2 \rceil$. In this paper we proved this consequence of the Conjecture (the bound (130)) just using the end of M .

³⁵For example, fundamental groups of finite volume, non-compact hyperbolic n -manifolds have parabolic abelian subgroups of rank $n-1$.

³⁶In fact, any d -complex is homotopy equivalent to a $2d$ -manifold by [10].

11.5. Group cohomology. Using duality, we can rephrase (127) as a vanishing result for low dimensional group cohomology: The long exact homology sequence, excision, and Lefschetz duality give

$$(132) \quad \overline{H}_*(\widetilde{M}_{<\epsilon}) \cong H_{*+1}(\widetilde{M}, \widetilde{M}_{<\epsilon}) \cong H_{*+1}(\widetilde{M}_{\geq\epsilon}, \partial) \cong H_c^{n-1-*}(\widetilde{M}_{\geq\epsilon}),$$

where H_c^* denotes cohomology with compact supports. Since the thick part $\widetilde{M}_{\geq\epsilon}$ is contractible and the Γ -action on it is cocompact, the right hand side of (132) is isomorphic to group cohomology with group ring coefficients $H^{n-1-*}(B\Gamma; \mathbb{Z}\Gamma)$. Putting this together with (127) gives

$$(133) \quad H^{<[n/2]}(B\Gamma; \mathbb{Z}\Gamma) = 0.$$

More generally, using (128) instead of (127) gives

$$(134) \quad H^{<\max\{n-r, [n/2]\}}(B\Gamma; \mathbb{Z}\Gamma) = 0,$$

which is stronger than (133) when r is small.

In the extreme case when $\text{gdim}(\Gamma) = [n/2]$, equation (133) implies the group cohomology $H^*(B\Gamma; \mathbb{Z}\Gamma)$ is concentrated in dimension $[n/2]$, so in this case Γ is an $[n/2]$ -dimensional duality group.

11.6. Boundries of fundamental groups are highly connected. If the group Γ has a Z -boundary³⁷, then (133) implies the space Z has no reduced Čech cohomology in dimensions $< [n/2] - 1$ (Proposition 1.5 in [5]).

11.7. Some groups that do not occur as fundamental groups $\pi_1 M$. Finally, we illustrate how the results of this paper can be used to see that some classes of CAT(0) groups do not occur as fundamental groups of M . Let L be a finite flag complex. It defines a *right angled Artin group*

$$A_L := \langle a_v, v \text{ vertex of } L \mid a_v a_w = a_w a_v \text{ if } v \text{ and } w \text{ are adjacent in } L \rangle.$$

The groups \mathbb{Z}^n and $F_2 \times \cdots \times F_2$ are special cases of A_L that are fundamental groups of nonpositively curved manifolds. However, we can use subsection 11.5 to see that many of the groups A_L cannot be fundamental groups of the manifolds M that we consider in this paper.³⁸ This is based on two things.

- If L is a closed manifold, or more generally if $H_{\dim L}(L; \mathbb{Z}/2) \neq 0$, then the dimension of any manifold homotopy equivalent to BA_L is $\geq 2 \cdot \text{gdim}(A_L)$. ([2]) So, if such an A_L is the fundamental group of M we must be in the extreme case $\dim M = 2 \cdot \text{gdim}(A_L)$.
- If L is a closed manifold, then A_L is a duality group if and only if L is a homology sphere. More generally, A_L is a duality group if and only if the complex L is *Cohen-Macaulay*.³⁹ ([7])

³⁷Gromov boundaries of hyperbolic groups and CAT(0) boundaries of CAT(0) groups are examples of Z -boundaries. The definition and more examples can be found in [5].

³⁸Recall that M is a complete, finite volume, Riemannian manifold of bounded non-positive curvature and it has no arbitrarily small geodesic loops.

³⁹This means the reduced homology $\overline{H}_*(L)$ is concentrated in dimension $\dim L$ and for any simplex $\sigma \subset L$, the reduced homology of its link $\overline{H}_*(\text{Link}_L(\sigma))$ is concentrated in dimension $\dim L - |\sigma| - 1$.

So, if L is a closed manifold that is not a homology sphere,⁴⁰ then A_L cannot be the fundamental group of M .

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⁴⁰Or, more generally, if $H_{\dim L}(L; \mathbb{Z}/2) \neq 0$ and L is not Cohen-Macaulay.